ECE 145C / 218C, notes set xx: Basic Analysis of Analog Circuits

Review Notes: not to be covered in lectures

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Review Notes

It is expected that ECE145C/218C students be relatively familiar with analog circuit design.
At UCSB: ECE137A/B, junior-level analog transistor circuit design.

Nevertheless, student backgrounds will vary.

These summary notes are taken from ECE137A/B.
As needed, please also refer to the ECE137A/B online notes

When possible, the notes are simplified. Mostly principles, fewer details

DC models and bias analysis

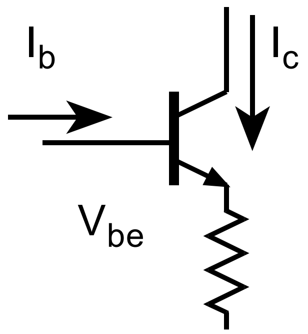
Large-Signal Model For Bias Analysis

Provided that $V_{ce} > 0$,

$I_c = I_s \exp(V_{be} / V_T)$ and $I_b = I_c / \beta$, where $V_T = kT / q$

...note that V_{be} is specified internal to the emitter resistance R_{ex}

The $I_e R_{ex}$ drop is significant for HBTs operating at current densities near that required for peak transistor bandwidth.

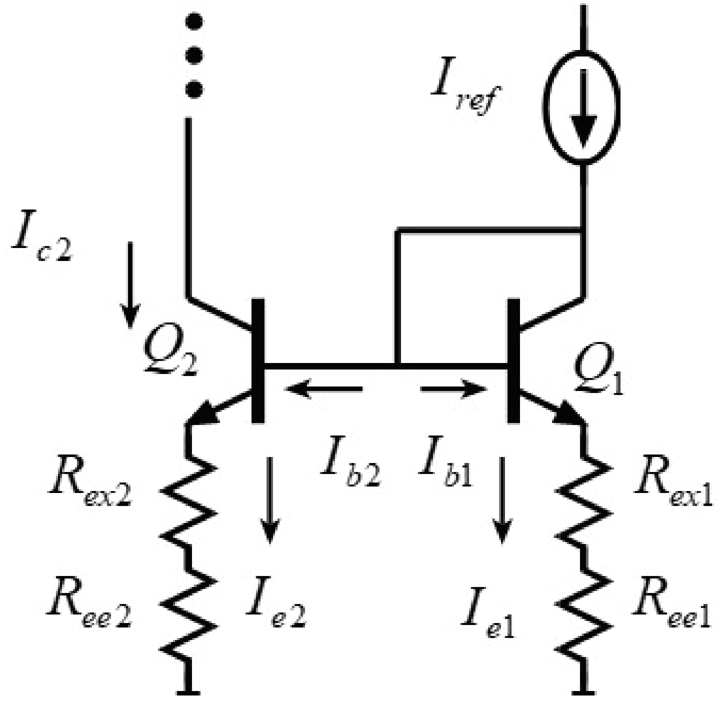


DC Bias Example: Current Mirror

We have $V_{be1}+I_{e1}(R_{ex1}+R_{ee1})=V_{be2}+I_{e2}(R_{ex2}+R_{ee2})$
and $V_{be1}=V_t\ln(I_{c1}/I_{s1})$, $V_{be2}=V_t\ln(I_{c2}/I_{s2})$

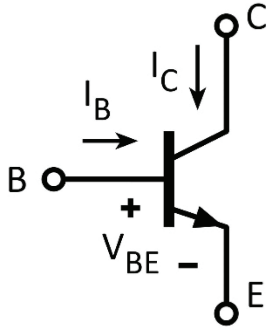
Assume that $\beta>>1$, $R_{ee2}=2R_{ee1}$
& assume that $A_{E1}=A_{E2}$ (A_E is the emitter area).

This implies $R_{ex1}=R_{ex2}/2$, and $I_{s1}=2I_{s2}$,
from which we find $I_{c2}=I_{c1}/2$



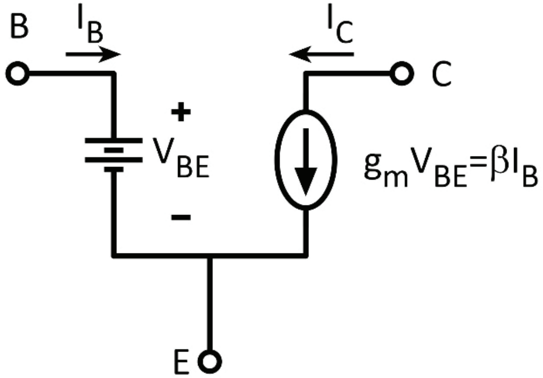
Simpler DC Model for Bias Analysis

It is often sufficient in bias analysis to ignore the variation of V_{be} with I_c and instead take $V_{be} = V_{be,on} = \varphi$.



$V_{be,on}$ depends upon current density and technology.
Biased at current densities within ~10% of peak bandwidth bias,

$$V_{be,on} = \varphi \sim \begin{cases} 0.9 \text{ V Modern Si/SiGe HBTs} \\ 0.8 \text{ V to } 0.9 \text{ V InGaAs/InP HBTs} \\ 1.4 \text{ V GaAs/GaInP HBTs} \end{cases}$$



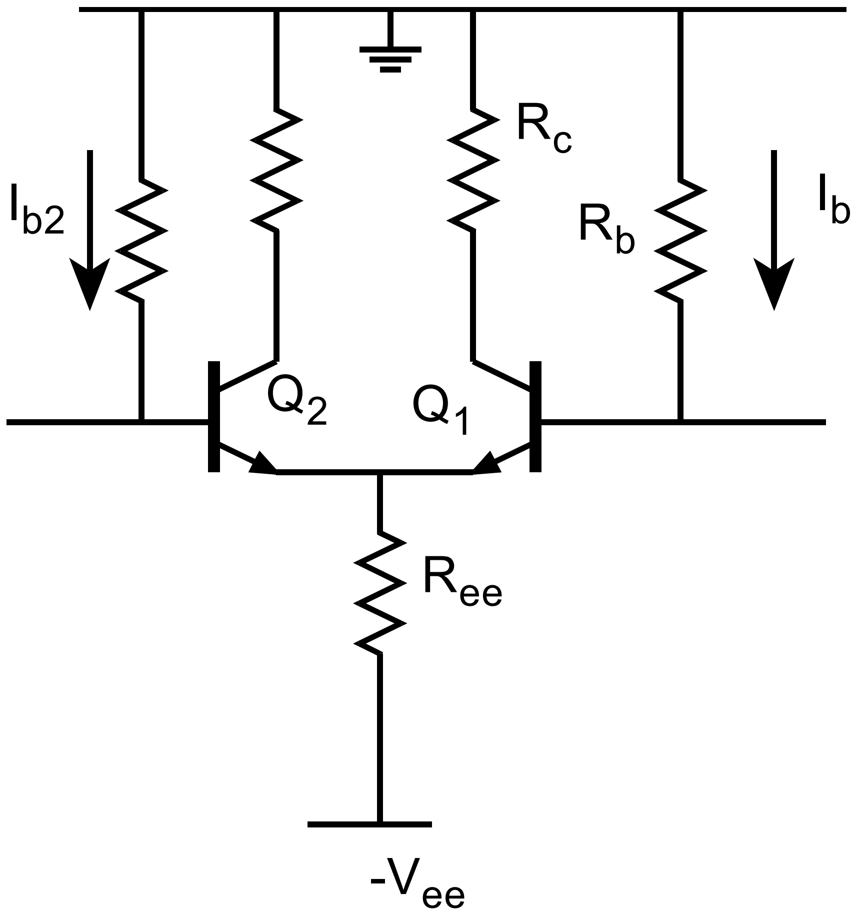
Simple DC Bias Example

If we neglect the $I_b R_b$ drops, then $V_{b1} = V_{b2} = 0$ Volts.

Approximate $V_{e1} = V_{e2} = -\varphi \cong -0.9$ V (SiGe).

$$I_{c1} + I_{c2} = 2I_{c1} = (-V_{ee} - 0.9V) / R_{ee}$$

$$I_{c1} = I_{c2} = (-V_{ee} - 0.9V) / 2R_{ee}$$



Efficiently Handling Base Currents In Bias Analysis

If $I_b R_b$ drop is singificant,

one can solve simultaneous equations:

$$I_{c1}+I_{c2}=2I_{c1}=(-V_{ee}-\varphi-I_{b1}R_b)/R_{ee}$$

where $I_{b1}=I_{c1}/\beta$,

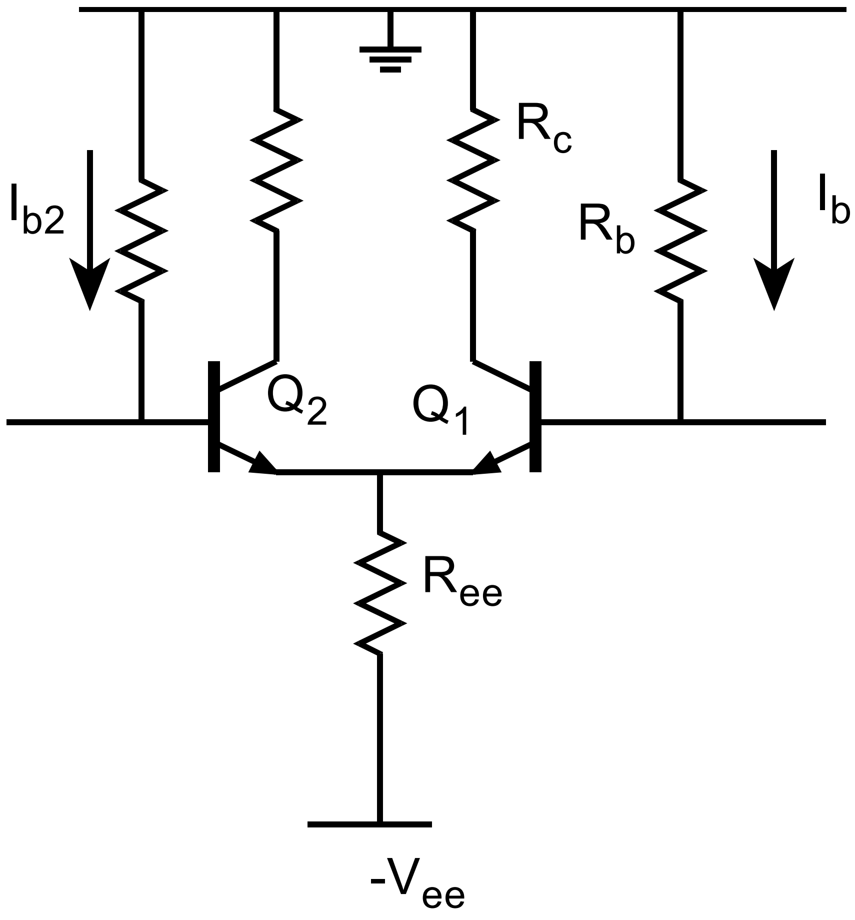
Quicker: find by iteration:

1) solve $I_{c1}=(-V_{ee}-\varphi)/2R_{ee}$

2) solve $I_{b1}\cong I_{c1}/\beta$

3) use this value of I_b to solve $I_{c1}=(-V_{ee}-\varphi-I_{b1}R_b)/2R_{ee}$

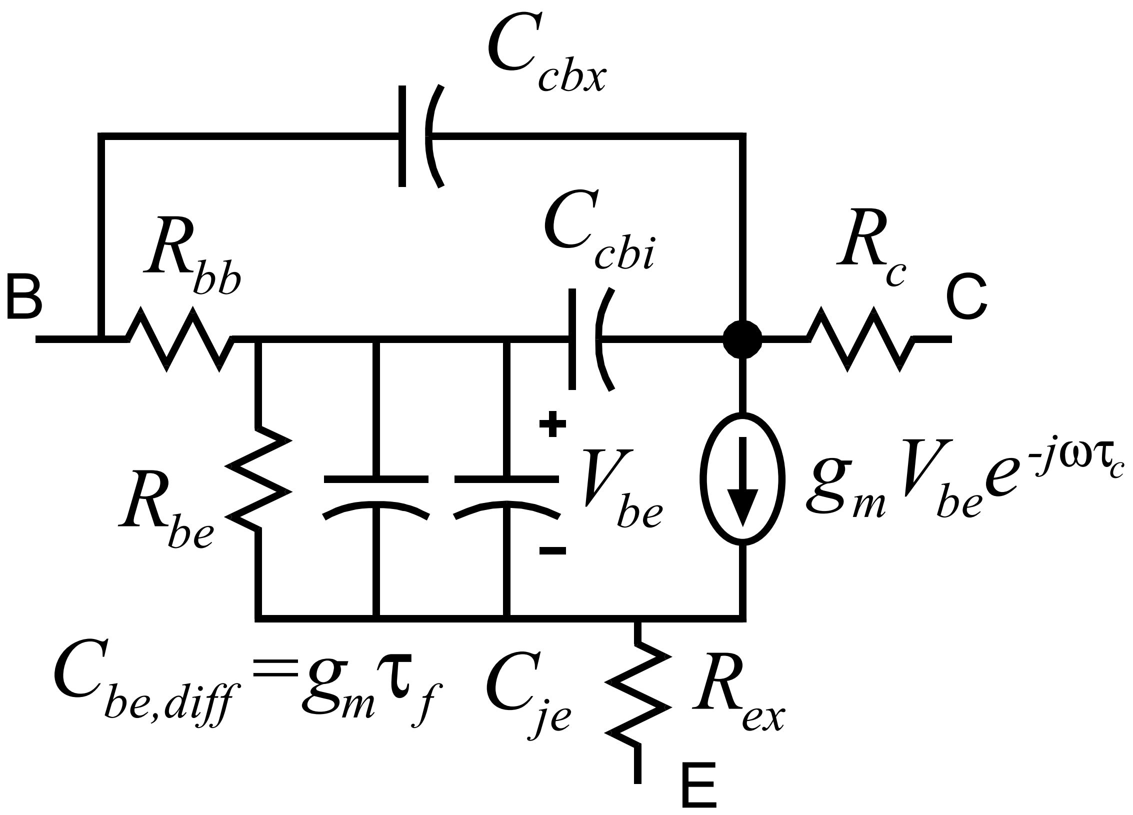
Works because any well-designed circuit has DC bias only weakly dependent upon β.



small-signal baseband analysis

Hybrid-π Bipolar Transistor Model

Fairly accurate model,
but too complex for quick hand analysis



$$\tau_f = \tau_b + \tau_c$$

$$R_{be} = \beta / g_m$$

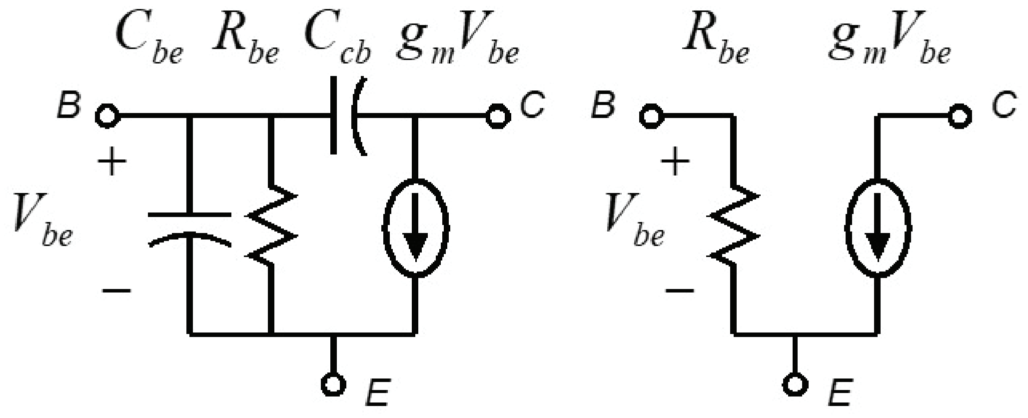
Oversimplified Model for Quick Hand Analysis

In most high-frequency circuits,
the node impedance is low
and R_{ce} is therefore negligible.

Neglecting R_{bb} in high-frequency analysis
is a poor approximation,
but is nevertheless common
in introductory treatments.

The simplified analyses that follow
use this oversimplified model.

These introductory treatments
will later be refined.



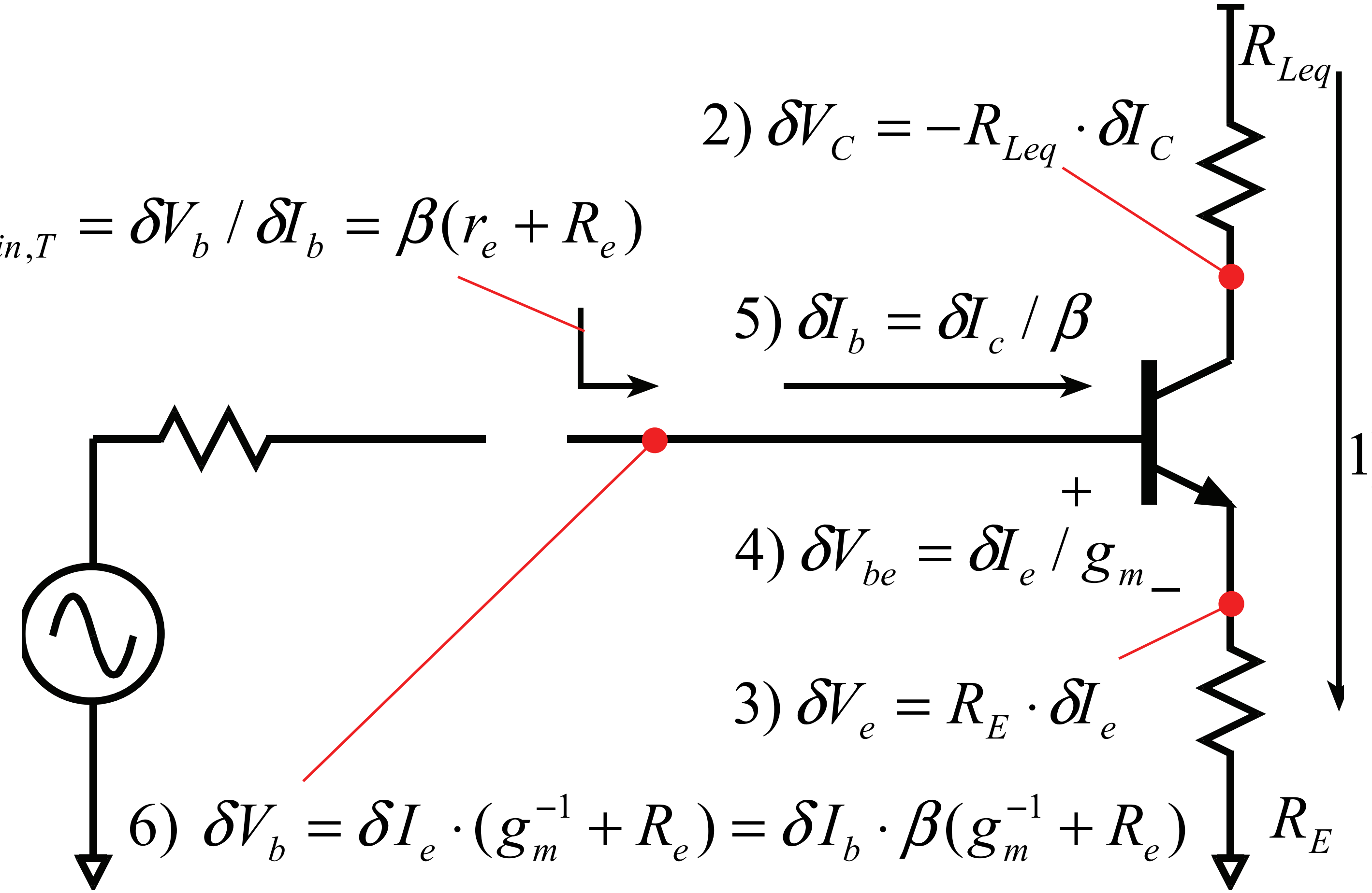
Common Emitter Stage: Basics

$$A_v = -R_{Leq} / (R_e + 1/g_m)$$

$$R_{in,transistor} = \beta(r_e + R_E)$$

This analysis assumes

$$R_{CE} >> R_{Leq}$$



$$1)\ \delta I_e \cong \delta I_c$$

$$2)\ \delta V_C = -R_{Leq} \cdot \delta I_C$$

$$3)\ \delta V_e = R_E \cdot \delta I_e$$

$$4)\ \delta V_{be} = \delta I_e / g_m$$

$$5)\ \delta I_b = \delta I_c / \beta$$

$$6)\ \delta V_b = \delta I_e \cdot (g_m^{-1} + R_e) = \delta I_b \cdot \beta(g_m^{-1} + R_e)$$

$$7)\ R_{in,T} = \delta V_b / \delta I_b = \beta(r_e + R_e)$$

$$8)\ \delta V_{out} / \delta V_{in} = \delta V_{out} / \delta V_B = -R_{Leq} / (R_e + 1/g_m)$$

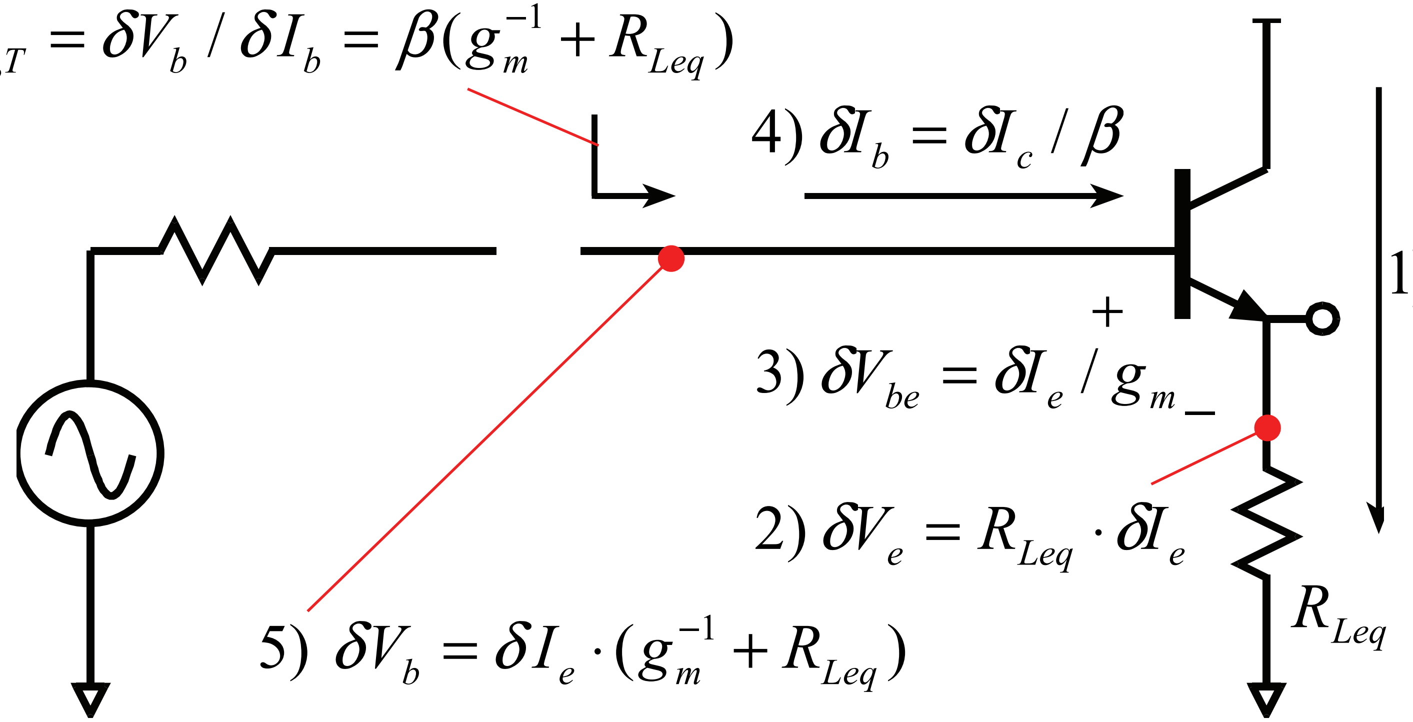
Emitter Follower Stage: Basics

$$A_v = R_{Leq} / (R_{Leq} + 1/g_m)$$

$$R_{in,transistor} = \beta(R_E + 1/g_m)$$

R_{CE} is included in R_{Leq}

6) $R_{in,T} = \delta V_b / \delta I_b = \beta(g_m^{-1} + R_{Leq})$



1) $\delta I_e \cong \delta I_c$

2) $\delta V_e = R_{Leq} \cdot \delta I_e$

3) $\delta V_{be} = \delta I_e / g_m$

4) $\delta I_b = \delta I_c / \beta$

5) $\delta V_b = \delta I_e \cdot (g_m^{-1} + R_{Leq})$

$= \delta I_b \cdot \beta(g_m^{-1} + R_{Leq})$

7) $\delta V_{out} / \delta V_{in} = \delta V_{out} / \delta V_E = R_{Leq} / (R_{Leq} + 1/g_m)$

Common-Base Stage: Basics

$$A_v = R_{Leq} / (g_m^{-1} + R_b / \beta)$$

$$R_{in,transistor} = g_m^{-1} + R_b / \beta$$

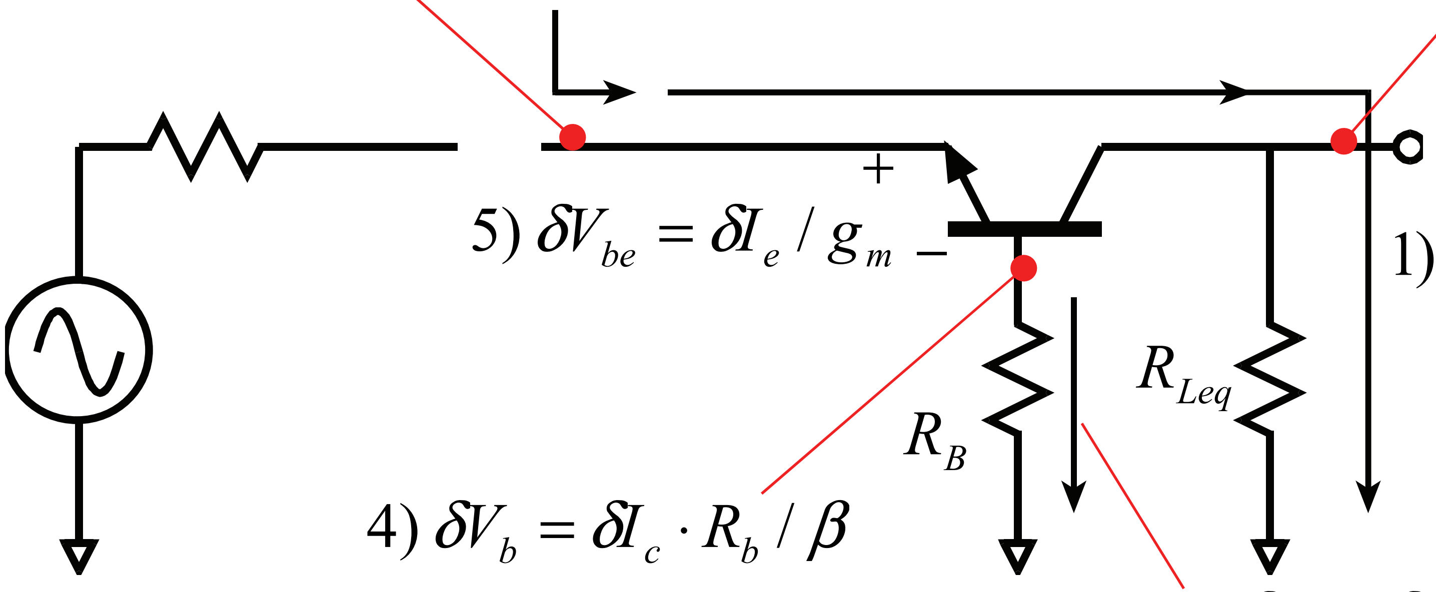
This analysis assumes

$$R_{CE} >> R_{Leq}$$

6) $\delta V_{in} = \delta I_e \cdot (r_e + R_b / \beta)$

2) $\delta V_{out} = R_{Leq} \cdot \delta I_C$

7) $R_{in,T} = \delta V_e / \delta I_e = r_e + R_B / \beta$



5) $\delta V_{be} = \delta I_e / g_m$

1) $\delta I_e \cong \delta I_c$

4) $\delta V_b = \delta I_c \cdot R_b / \beta$

3) $\delta I_b = \delta I_c / \beta$

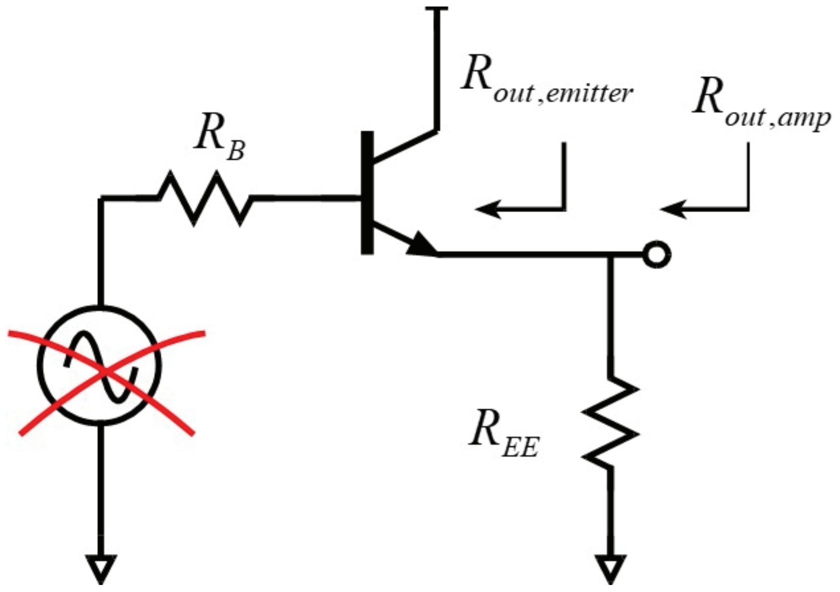
7) $\delta V_{out} / \delta V_{in} = R_{Leq} / (r_e + R_b / \beta)$

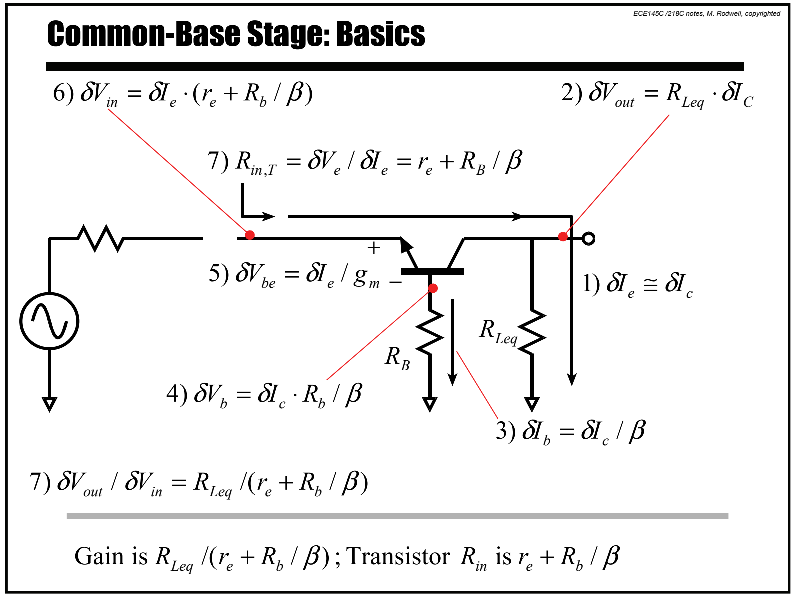
Emitter Follower Output Impedance

finding emitter follower output impedance
is the same calculation as
finding common-base input impedance

$$R_{out,emitter} = 1/g_m + R_B/\beta$$

$$R_{out,amp} = R_{out,emitter} \| R_{EE}$$

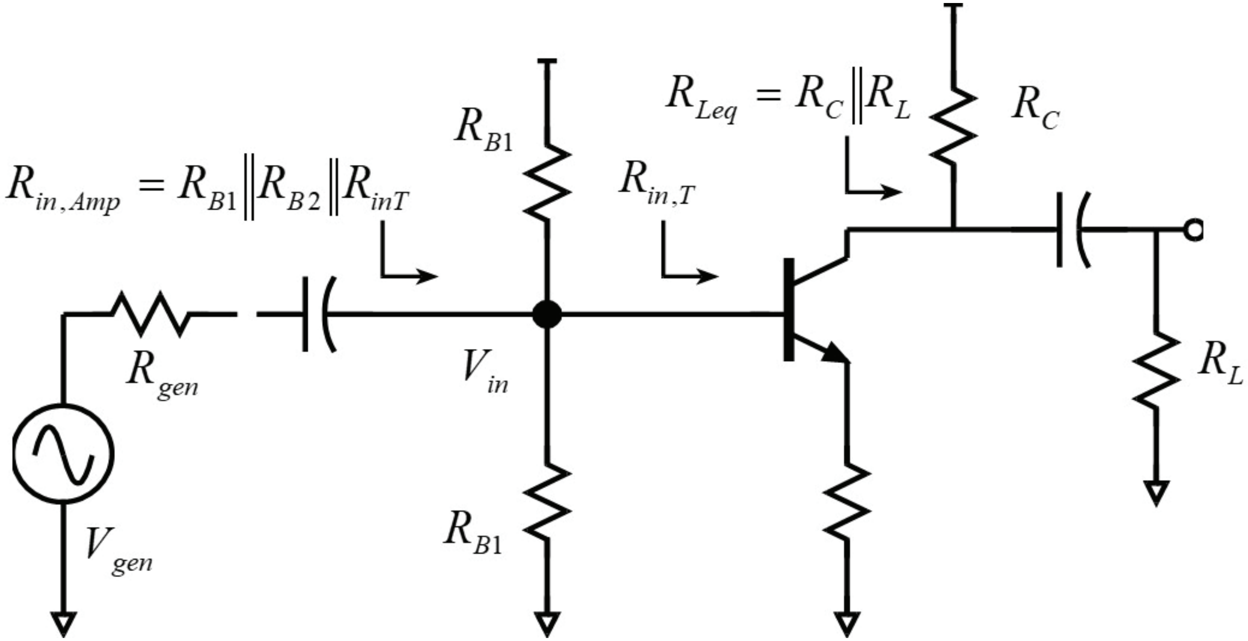




Including Bias Circuit Resistances

These are (trivially) added in parallel with the transistor terminal impedances to determine the net circuit impedances.

From which,
$V_{in} / V_{gen} = R_{in,amp} / (R_{in,amp} + R_{gen})$,
etc.

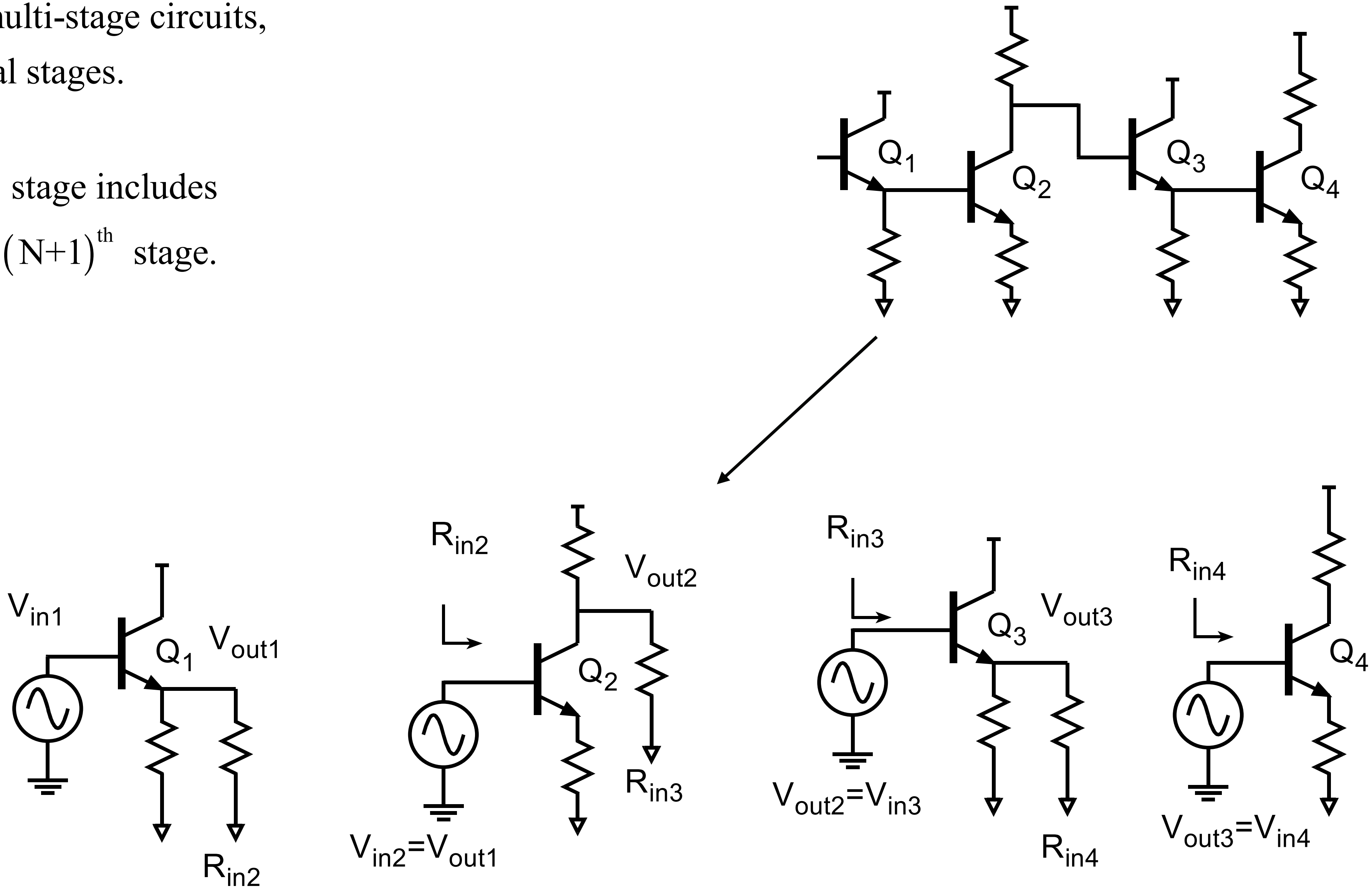


Baseband Analysis Of Multistage Circuits

For baseband analysis of multi-stage circuits, simply break into individual stages.

Load impedance of the N^{th} stage includes the input impedance of the $(N+1)^{th}$ stage.

Analysis is then trivial.



Small-Signal
Transfer Function Analysis

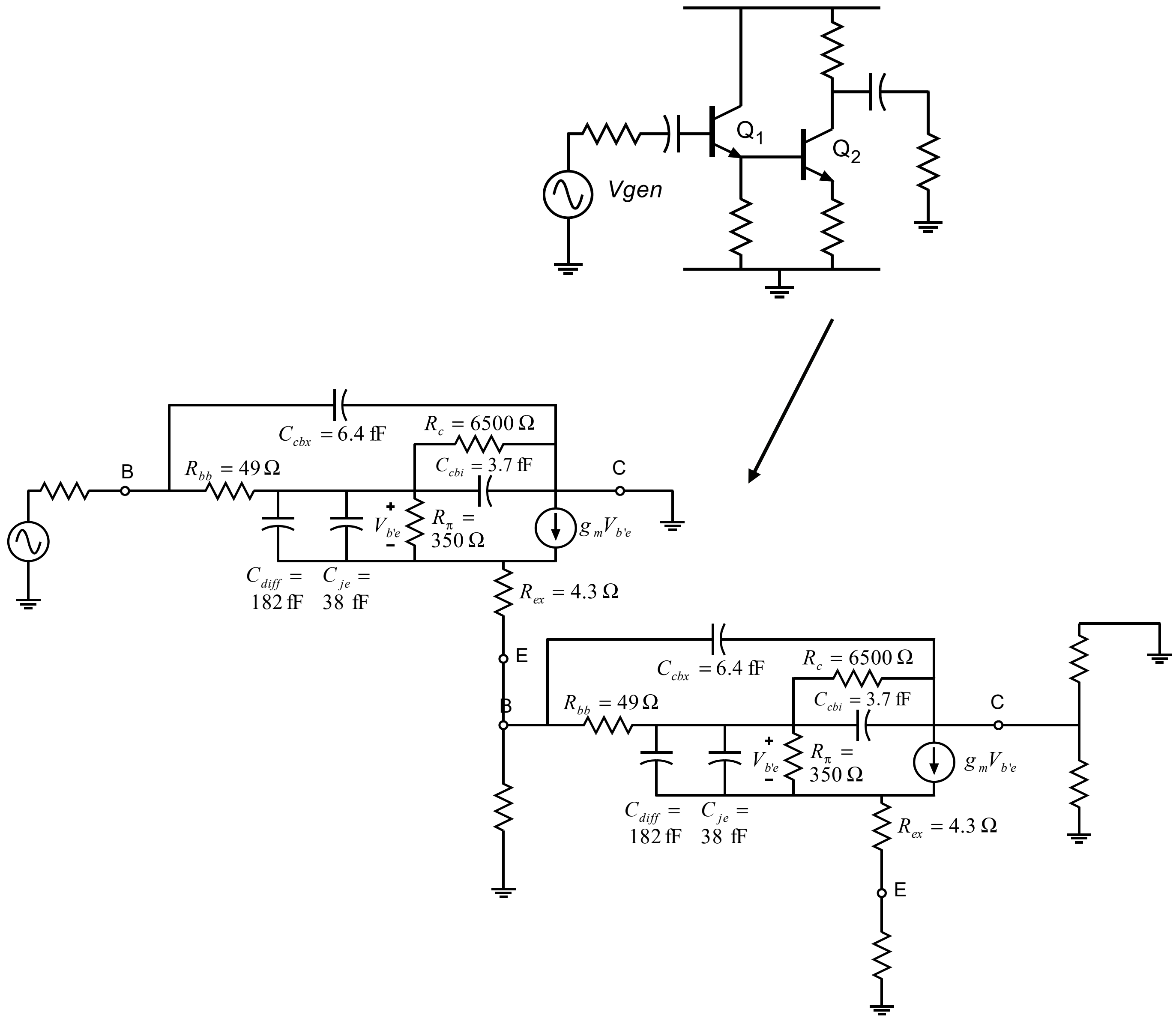
High-Frequency Analysis: The General Problem

Cannot separate multi-stage circuits
invididual stages, to analyze stage-by-stage.

Computing transfer function is thus difficult.

Method #1: nodal analysis:
accurate, general, tedious.

Method #2: method of time constants:
accurate,
limited applicability,
quick & intuitive



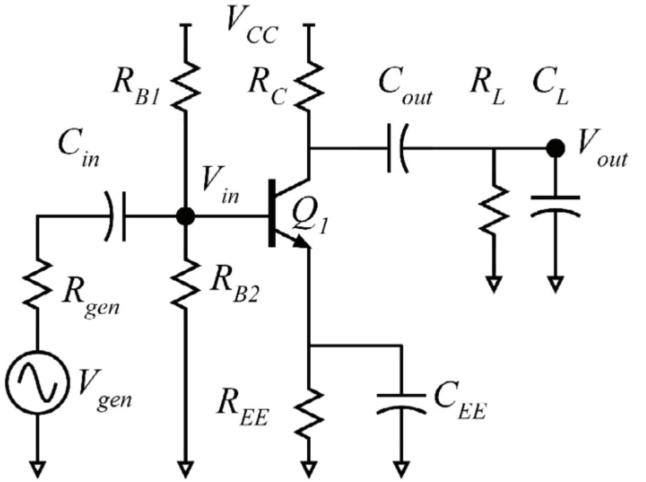
Nodal analysis

(from ECE137B notes)

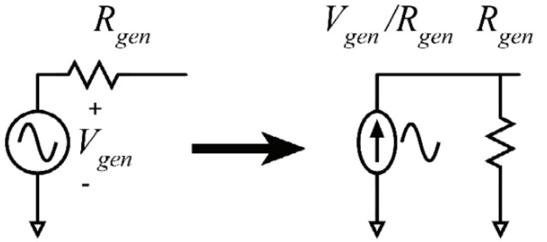
Common-emitter amplifier

This lecture: consider only high-frequency response:

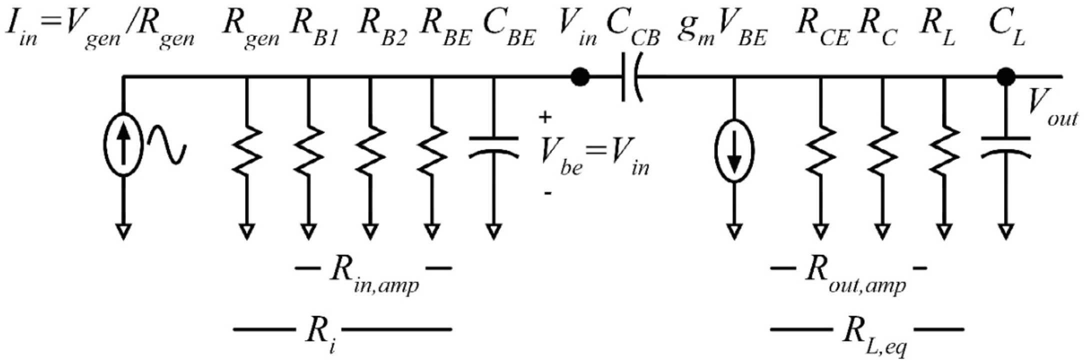
$\Rightarrow C_{in} = C_{out} = C_{EE} = \infty$ Farads



Save effort by using Norton, not Thevenin, generator model.



We now have the high-frequency equivalent circuit to the right $\Rightarrow$



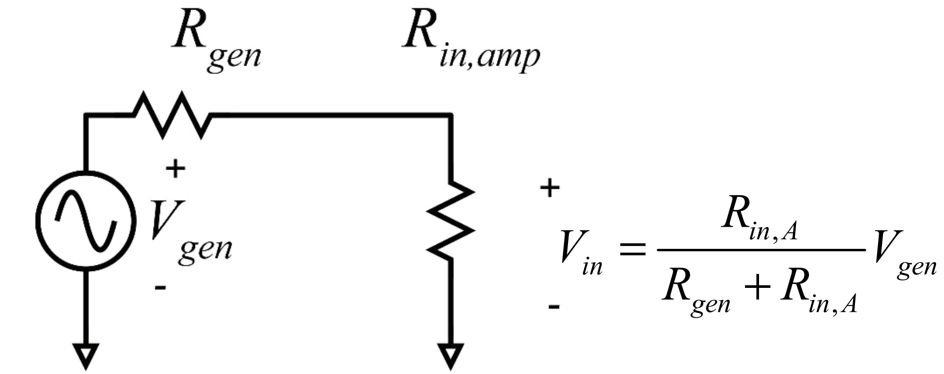
Aside: setting up a relationship to use later

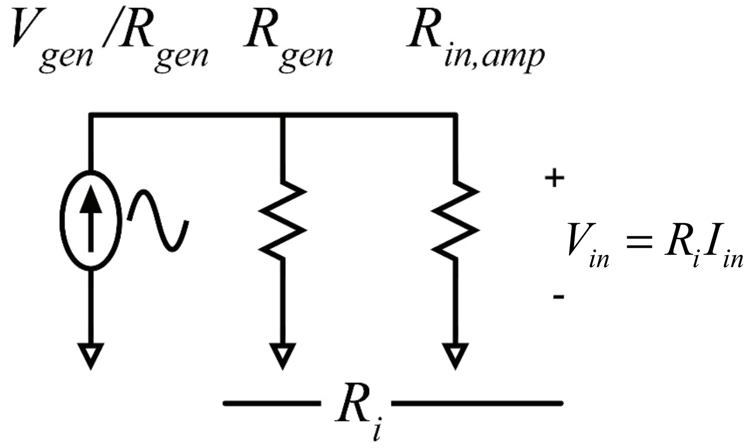
$$V_{in} = R_i I_{in} = \left(R_{gen} \parallel R_{in,A}\right)\left(\frac{V_{gen}}{R_{gen}}\right)$$

$$= \frac{R_{gen} R_{in,A}}{R_{gen} + R_{in,A}} \frac{V_{gen}}{R_{gen}} = \frac{R_{in,A}}{R_{gen} + R_{in,A}} V_{gen}$$

Learn to recognize that

$$R_i I_{in} = \frac{R_{in,A}}{R_{gen} + R_{in,A}} V_{gen}$$



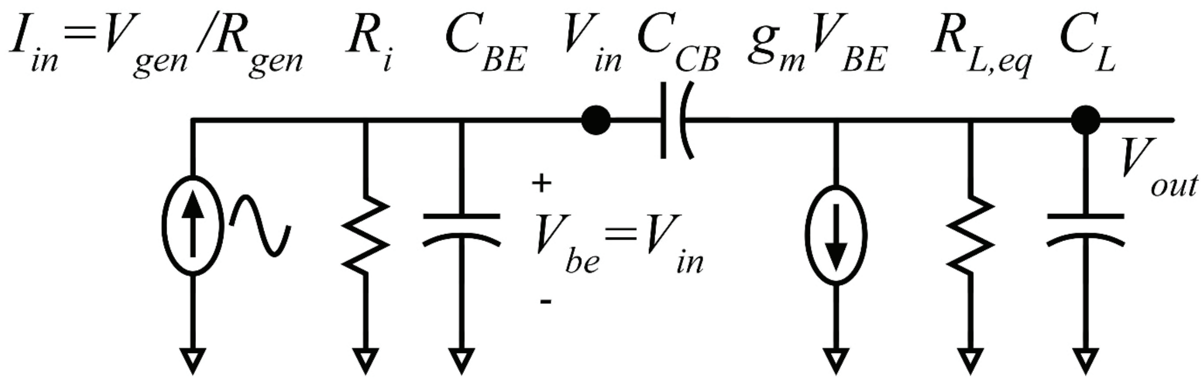


Simplifying the circuit

We are solving this problem for *2 reasons*

1) to get the answer

2) to review how solve circuit transfer functions, poles, zeros.

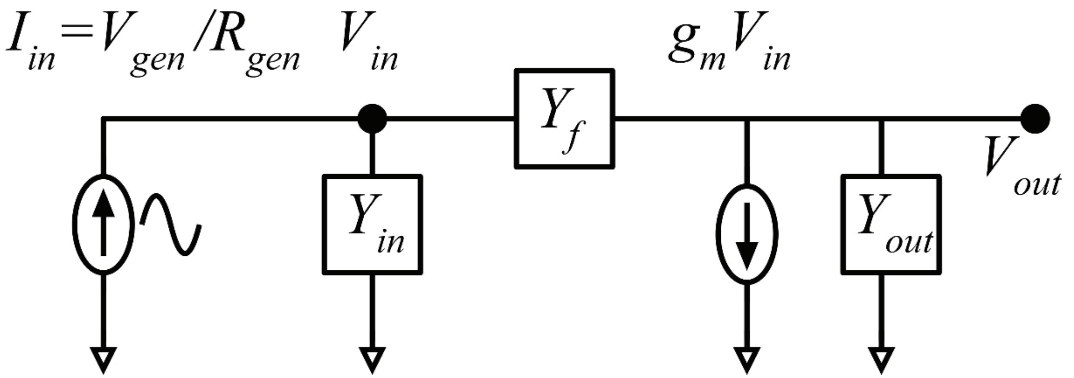


Problem is easier if we write

$$Y_{in}=1/R_i+sC_{be}=G_i+sC_{be}$$

$$Y_L=1/R_{L,eq}+sC_L=G_{L,eq}+sC_L$$

$$Y_f=sC_{cb}$$



Nodal Analysis: why we learn this.

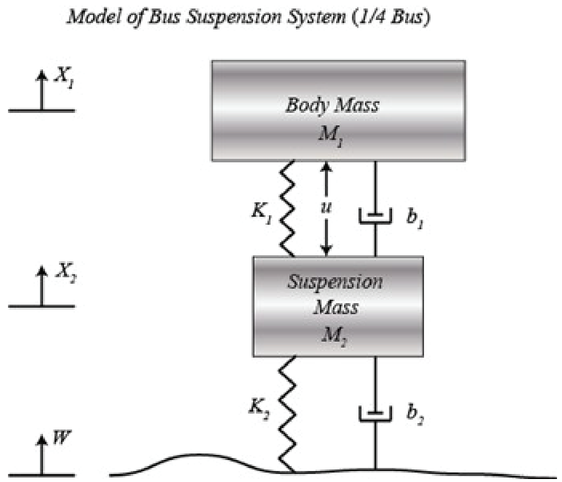
Please note: only on this note set will I show you all steps for nodal analysis. Please review your sophomore circuits analysis.

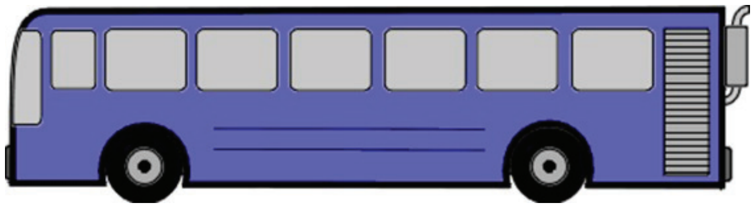
Though both nodal & mesh analysis are taught in the sophomore year, nodal analsis is usually easier and quicker.

In a circuit with N unknown node voltages, nodal analysis will always give you the N *linearly independent* equations you need to solve for these unknowns.

Nodal analysis: write $\Sigma I = 0$ at each node for which you do not know the node voltage.

You can use very similar methods to compute the dynamics of mechanical systems: acoustics, cars, planes, robotics, control systems





https://ctms.engin.umich.edu/CTMS/index.php?example=Suspension&section=SystemModeling

Nodal Analysis: setting up equations

$$\Sigma I = 0 \ @ \ V_{in} \ :$$

$$-I_{in} + V_{in}Y_{in} + (V_{in} - V_{out})Y_f = 0$$

$$\Rightarrow (Y_{in} + Y_f)V_{in} + (-Y_f)V_{out} = I_{in}$$

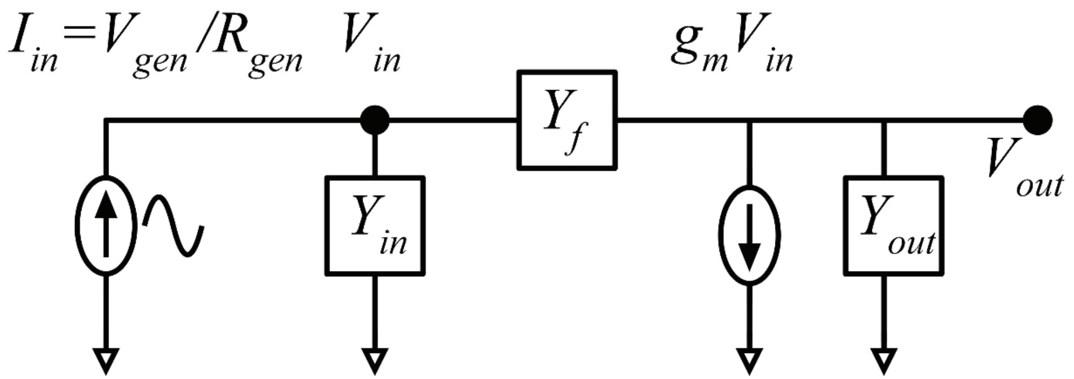
$$\Sigma I = 0 \ @ \ V_{out} \ :$$

$$g_m V_{in} + (V_{out} - V_{in})Y_f + V_{out}Y_L = 0$$

$$\Rightarrow (g_m - Y_f)V_{in} + (Y_L + Y_f)V_{out} = 0$$

this can be written in matrix form:

$$\begin{bmatrix} Y_{in} + Y_f & -Y_f \\ g_m - Y_f & Y_L + Y_f \end{bmatrix} \begin{bmatrix} V_{in} \\ V_{out} \end{bmatrix} = \begin{bmatrix} I_{in} \\ 0 \end{bmatrix}$$



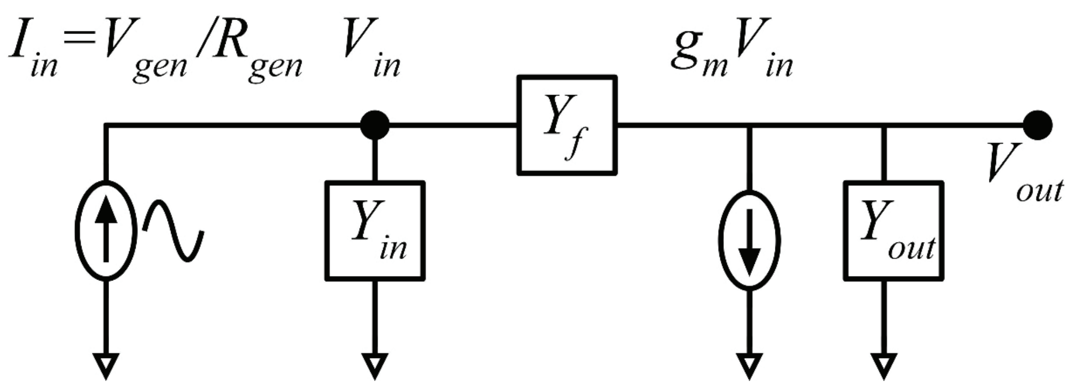
Systems of equations can be solved many ways.

Use your favorite method.

One method (not a particularly efficient one) is Cramer's rule.

Nodal Analysis: solving equations (1)

$$\begin{bmatrix} Y_{in}+Y_f & -Y_f \\ g_m-Y_f & Y_L+Y_f \end{bmatrix}\begin{bmatrix} V_{in} \\ V_{out} \end{bmatrix}=\begin{bmatrix} I_{in} \\ 0 \end{bmatrix}$$



Cramer's rule:

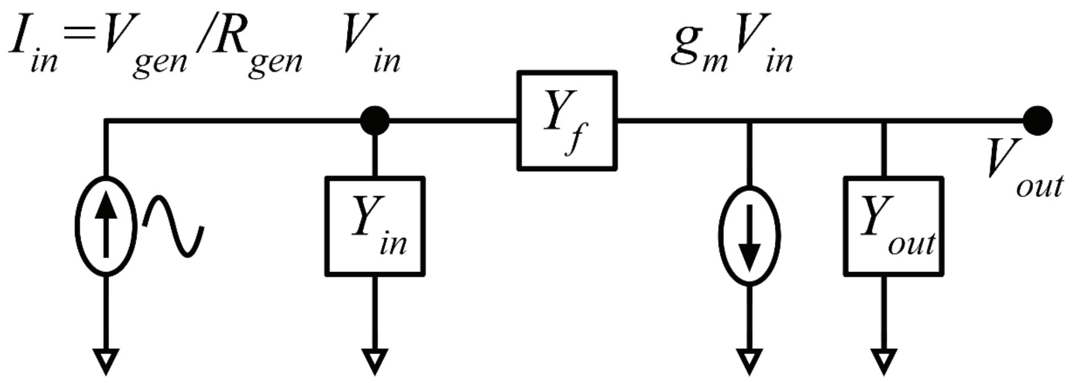
$$V_{out}=\frac{N}{D}=\frac{\begin{vmatrix} Y_{in}+Y_f & I_{in} \\ g_m-Y_f & 0 \end{vmatrix}}{\begin{vmatrix} Y_{in}+Y_f & -Y_f \\ g_m-Y_f & Y_L+Y_f \end{vmatrix}}$$

$$N=\begin{vmatrix} Y_{in}+Y_f & I_{in} \\ g_m-Y_f & 0 \end{vmatrix}=(Y_{in}+Y_f)(0)-(I_{in})(g_m-Y_f)=-(I_{in})(g_m-sC_{cb})$$

Put in dimensionless polynomial form:

$$N=-I_{in}(1-sC_{cb}/g_m)=-g_mI_{in}(1-sC_{cb}/g_m)$$
$$=-g_mI_{in}(1+b_1s) \text{ where } b_1=-C_{cb}/g_m$$

Nodal Analysis: solving equations (2)



$$D=\begin{vmatrix} Y_{in}+Y_f & -Y_f \\ g_m-Y_f & Y_L+Y_f \end{vmatrix}=(Y_{in}+Y_f)(Y_L+Y_f)-(-Y_f)(g_m-Y_f)$$

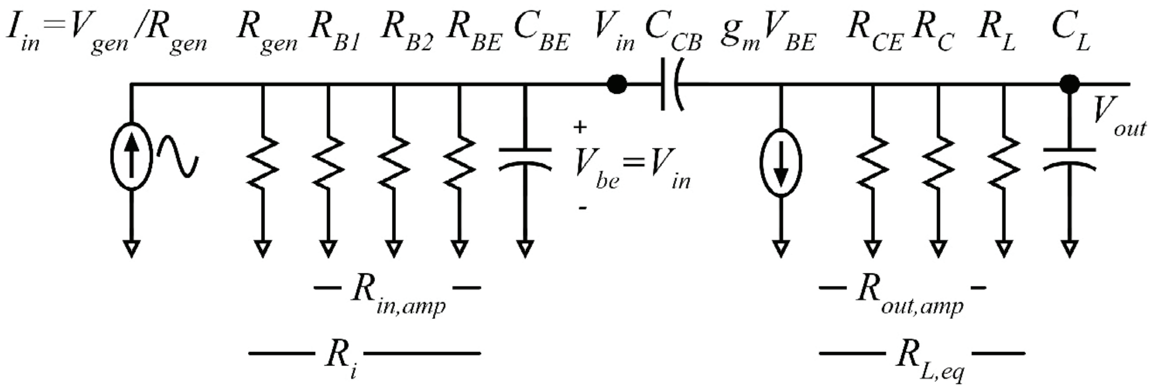
$$=Y_{in}Y_L+Y_{in}Y_f+Y_fY_L+Y_f^2+g_mY_f-Y_f^2$$

$$=Y_{in}Y_L+Y_{in}Y_f+Y_fY_L+g_mY_f$$

$$=(G_i+sC_{be})(G_{L,eq}+sC_L)+(G_i+sC_{be})sC_{cb}+sC_{cb}(G_{L,eq}+sC_L)+g_msC_{cb}$$

As you multiply out, organize by powers of s, i.e. s^0, s^1, and s^2 :

$$D=G_iG_{L,eq}$$
$$+s\left(G_iC_L+G_{L,eq}C_{be}+G_iC_{cb}+G_{L,eq}C_{cb}+g_mC_{cb}\right)$$
$$+s^2\left(C_{cb}C_{be}+C_{cb}C_L+C_{be}C_L\right)$$



Nodal Analysis: solving equations (3)

We must put D into dimensionless polynomial form:

Do this by dividing each term in D by $G_i G_{Leq} = 1/R_i R_{Leq}$:

$$D = (R_i R_{L,eq})^{-1}(1 + a_1 s + a_2 s^2)$$

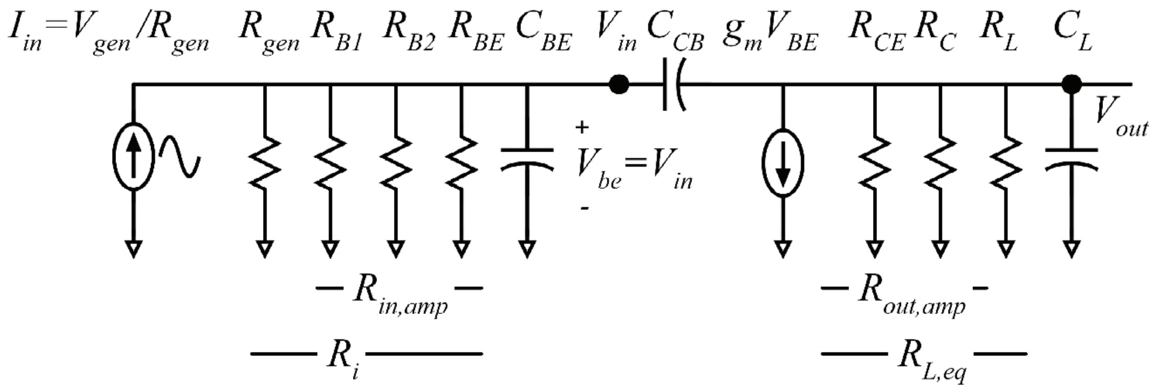
where

$$a_1 = R_{L,eq}C_{cb} + R_{Leq}C_L + R_i C_{be} + R_i C_{cb} + R_i R_{L,eq} g_m C_{cb} =$$

$$= R_i C_{be} + R_{L,eq}C_{cb} + R_{L,eq}C_L + R_i(1 + g_m R_{L,eq})C_{cb}$$

and

$$a_2 = R_i R_{Leq}\left(C_{cb}C_{be} + C_{cb}C_L + C_{be}C_L\right)$$



$$V_{out} = \frac{N}{D} = \frac{-g_m I_{in}(1 + b_1 s)}{(R_i R_{L,eq})^{-1}(1 + a_1 s + a_2 s^2)} = -g_m I_{in} R_i R_{L,eq} \frac{1 + b_1 s}{1 + a_1 s + a_2 s^2}$$

but $I_{in} R_i = V_{gen} \dfrac{R_{in,A}}{R_{in,A} + R_{gen}}$ so:

$$\frac{V_{out}}{V_{gen}} = -g_m R_{L,eq} \frac{R_{in,A}}{R_{in,A} + R_{gen}} \frac{1 + b_1 s}{1 + a_1 s + a_2 s^2}$$

Nodal Analysis: solution

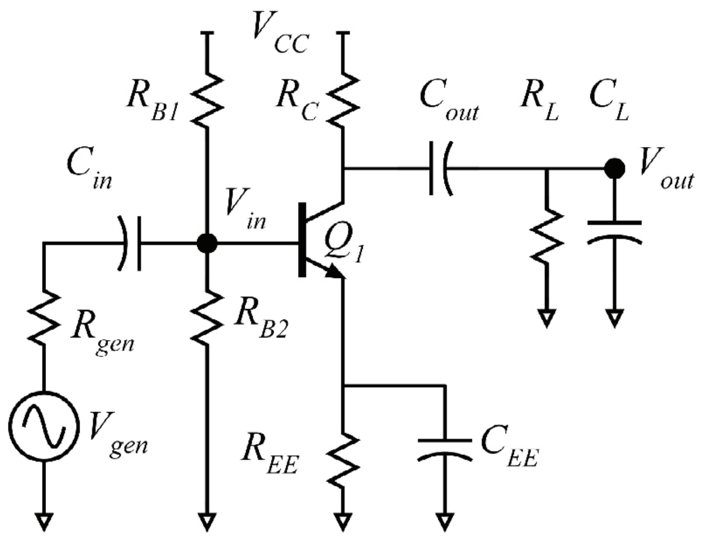
$$\frac{V_{out}(s)}{V_{gen}(s)} = -g_m R_{L,eq} \frac{R_{in,A}}{R_{in,A}+R_{gen}} \frac{1+b_1 s}{1+a_1 s + a_2 s^2}$$

where

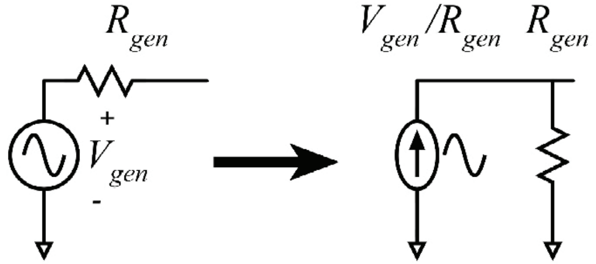
$$a_1 = R_i C_{be} + R_{L,eq} C_{cb} + R_{L,eq} C_L + R_i (1+g_m R_{L,eq}) C_{cb}$$

$$a_2 = R_i R_{Leq} \left(C_{cb} C_{be} + C_{cb} C_L + C_{be} C_L \right)$$

$$b_1 = -C_{cb} / g_m$$

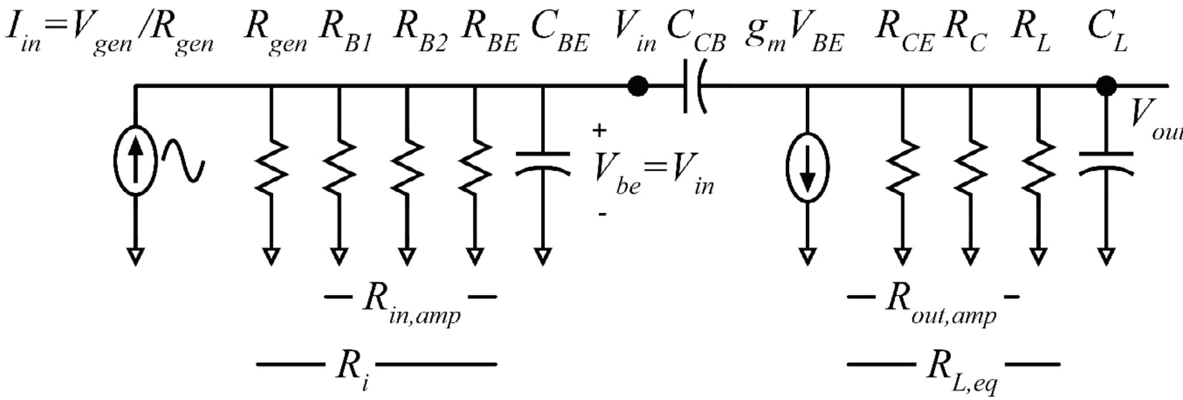


This has been slow because all steps have been shown (just this once).



Answer has both the DC gain and the frequency response.

Since we know easier methods (from 137A) to find DCgain we will often dorp constants during the AC analysis.



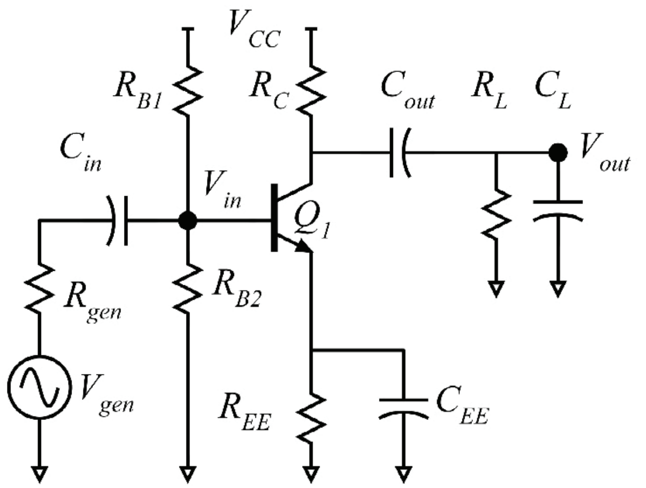
The answer, though complicated, makes perfect sense, and you will become familiar with it.

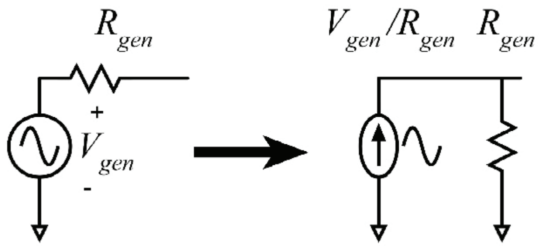
Mid-band gain and Frequency response

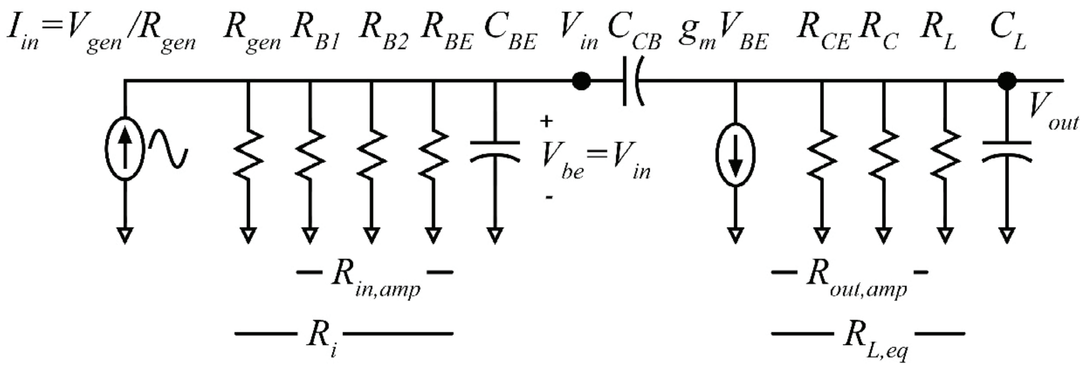
$$\frac{V_{out}(s)}{V_{gen}(s)} = -g_m R_{L,eq} \frac{R_{in,A}}{R_{in,A}+R_{gen}} \frac{1+b_1 s}{1+a_1 s+a_2 s^2} = \left.\frac{V_{out}}{V_{gen}}\right|_{mid-band} \cdot \frac{1+b_1 s}{1+a_1 s+a_2 s^2}$$

where

$$\left.\frac{V_{out}}{V_{gen}}\right|_{mid-band} = \text{ECE137A answer} = -g_m R_{L,eq} \frac{R_{in,A}}{R_{in,A}+R_{gen}}$$







Finding the poles and zeros.

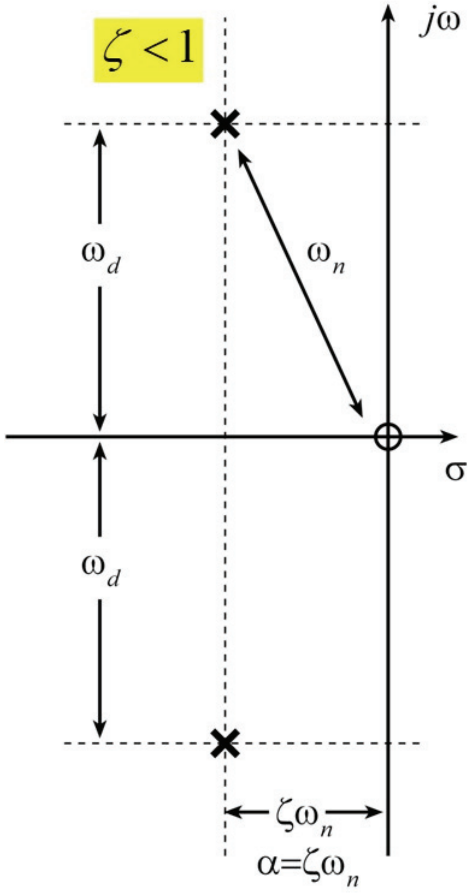
$$\frac{V_{out}(s)}{V_{gen}(s)} = \left.\frac{V_{out}}{V_{gen}}\right|_{mid-band} \cdot \frac{1+b_1 s}{1+a_1 s+a_2 s^2}$$

In general, we can write

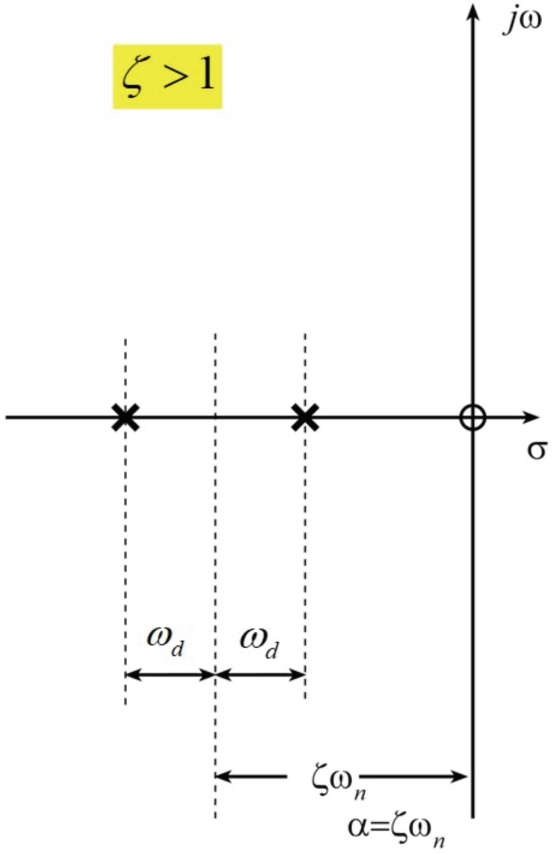
$$1+a_1 s+a_2 s^2 = 1+s(2\zeta/\omega_n)+s^2/\omega_n^2$$

$$\omega_n = a_2^{-1/2} \text{ and } \zeta = a_1\omega_n/2 = a_1 a_2^{-1/2}/2$$

Pole locations are then found as to the right.



$$\omega_d = \omega_n \cdot \sqrt{1-\zeta^2}$$



$$\omega_d = \omega_n \cdot \sqrt{\zeta^2 - 1}$$

But, if $a_2/a_1 << a_1$, there is a simpler way of solving this

Separated pole approximation

$$\frac{V_{out}(s)}{V_{gen}(s)} = \left.\frac{V_{out}}{V_{gen}}\right|_{mid-band} \cdot \frac{1+b_1 s}{1+a_1 s+a_2 s^2}$$

Consider a system with two real poles

$(1+s\tau_1)(1+s\tau_2) = 1+s(\tau_1+\tau_2)+s^2\tau_1\tau_2$

Now suppose than $\tau_1 >> \tau_2$:

$(1+s\tau_1)(1+s\tau_2) \cong 1+s\tau_1+s^2\tau_1\tau_2$

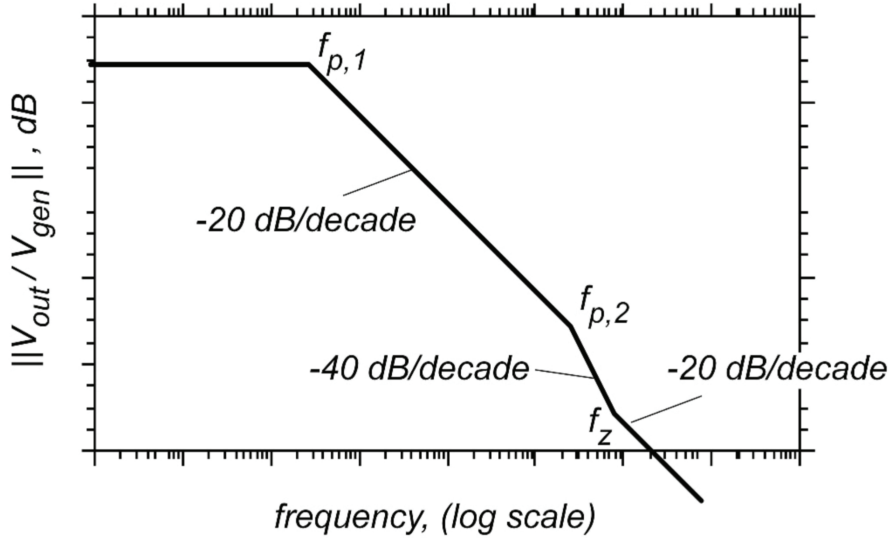
This means we can approximately factor $1+a_1 s+a_2 s^2$:

$1+a_1 s+a_2 s^2 \cong (1+a_1 s)(1+(a_2/a_1)s)$ *iff* $a_1 >> a_2/a_1$

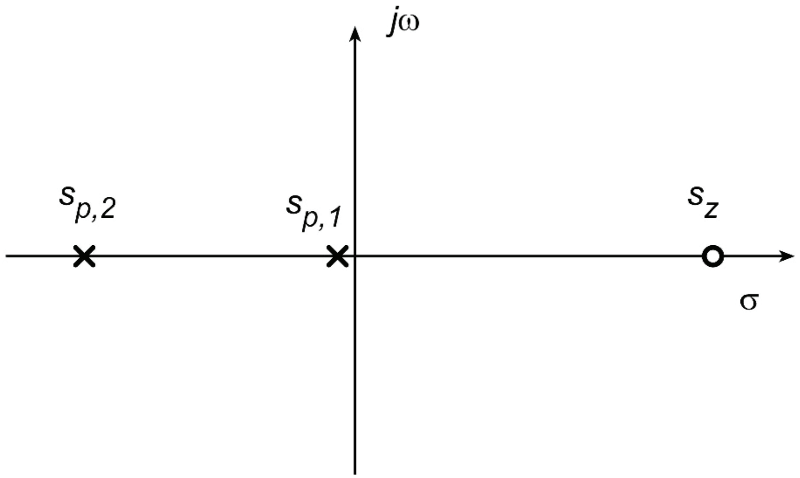
$$\frac{V_{out}(s)}{V_{gen}(s)} \cong \left.\frac{V_{out}}{V_{gen}}\right|_{mid-band} \cdot \frac{1+b_1 s}{(1+a_1 s)(1+(a_2/a_1)s)} \text{ *iff* } a_1 >> a_2/a_1$$

dominant pole: $f_{p1} \cong 1/2\pi a_1$; secondary pole: $f_{p2} \cong 1/2\pi(a_2/a_1)$

Bode plot



Pole-zero constellation -20 dB/decade



Frequency response of common-source stage

Clearly the same problem, except for different notation

$$\frac{V_{out}(s)}{V_{gen}(s)} = \left.\frac{V_{out}}{V_{gen}}\right|_{mid-band} \cdot \frac{1+b_1 s}{1+a_1 s + a_2 s^2}$$

where

$$a_1 = R_i C_{gs} + R_{L,eq} C_{gd} + R_{L,eq} C_L + R_i (1 + g_m R_{L,eq}) C_{gd}$$

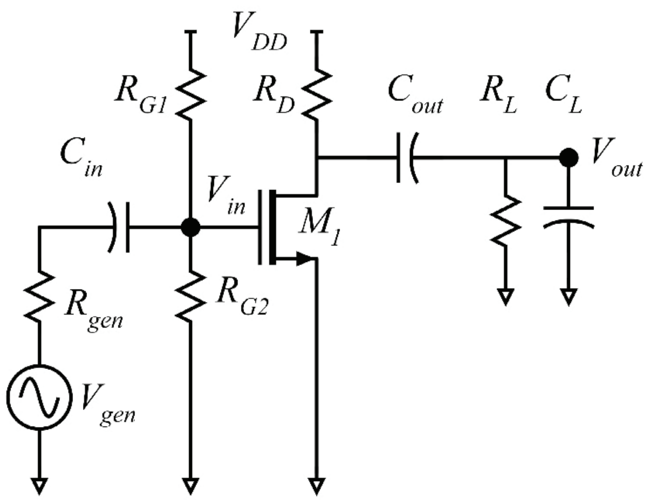
$$a_2 = R_i R_{Leq} \left(C_{gs} C_{gd} + C_{gs} C_L + C_{gd} C_L \right)$$

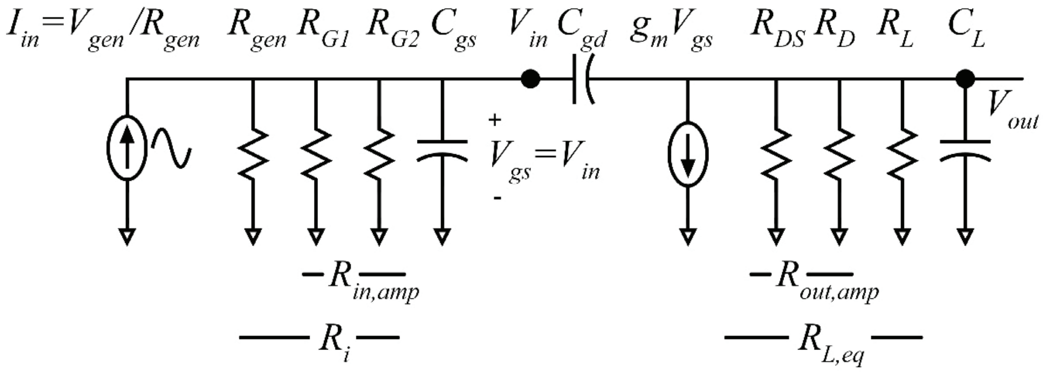
$$b_1 = -C_{gd} / g_m \rightarrow 1/2\pi f_z = -C_{gd} / g_m$$

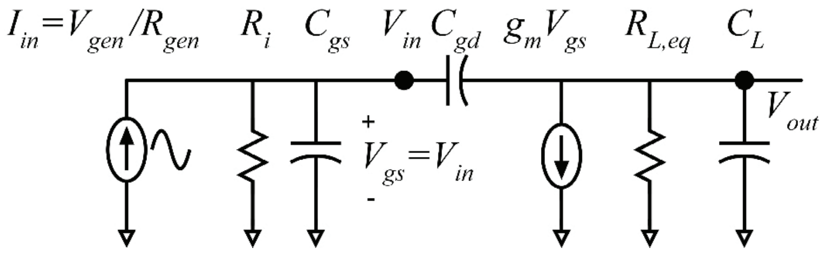
...and if we can use the separated pole approximation:

$$1/2\pi f_{p1} \cong a_1 = R_i C_{gs} + R_{L,eq} C_{gd} + R_{L,eq} C_L + R_i (1 + g_m R_{L,eq}) C_{gd}$$

$$1/2\pi f_{p2} \cong \frac{a_2}{a_1} = \frac{R_i R_{Leq} \left(C_{gs} C_{gd} + C_{gs} C_L + C_{gd} C_L \right)}{RR_i C_{gs} + R_{L,eq} C_{gd} + R_{L,eq} C_L + R_i (1 + g_m R_{L,eq}) C_{gd}}$$







Method of Time Constants

Methods of computing circuit transfer functions

General circuit transfer function: $\dfrac{v_{out}(s)}{v_{gen}(s)} = \left.\dfrac{v_{out}}{v_{gen}}\right|_{\text{mid-band}} \cdot \dfrac{1 + b_1 s^1 + b_2 s^2 + b_3 s^3 + \ldots}{1 + a_1 s^1 + a_2 s^2 + a_3 s^3 + \ldots}$

Nodal analysis:
hard to compute for complex circuits
answers often hard to interpret

Simulations (CAD)
Equivalent to randomly building & testing
zero insights.
for verification, not for synthesis or invention

Method of time constants
Exact...for the limited information it gives
easy and fast
answers often easily understood and interpretable

MOTC for inductor-free circuits

The MOTC variant we will study
is restricted to inductor-free circuits.

Frequency response (poles and zeros)
due only to capacitors.

This is sufficient for our purposes here.

Other MOTC variants allow analysis of
circuits with both capacitors and inductors.

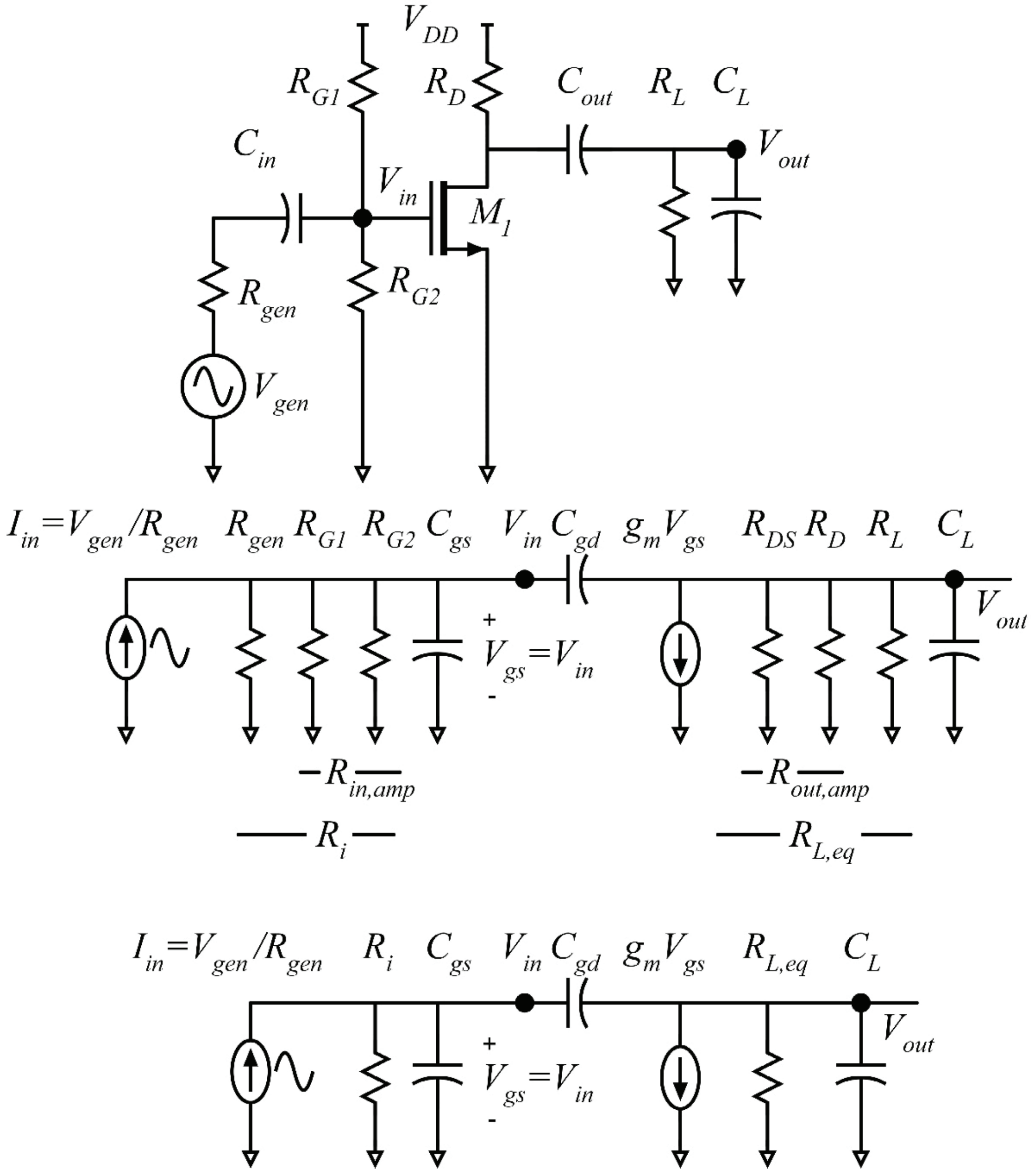
Method of Time Constants: Example Circuit

Example circuit:

common-source amplifier

we have already analyzed

by nodal analysis

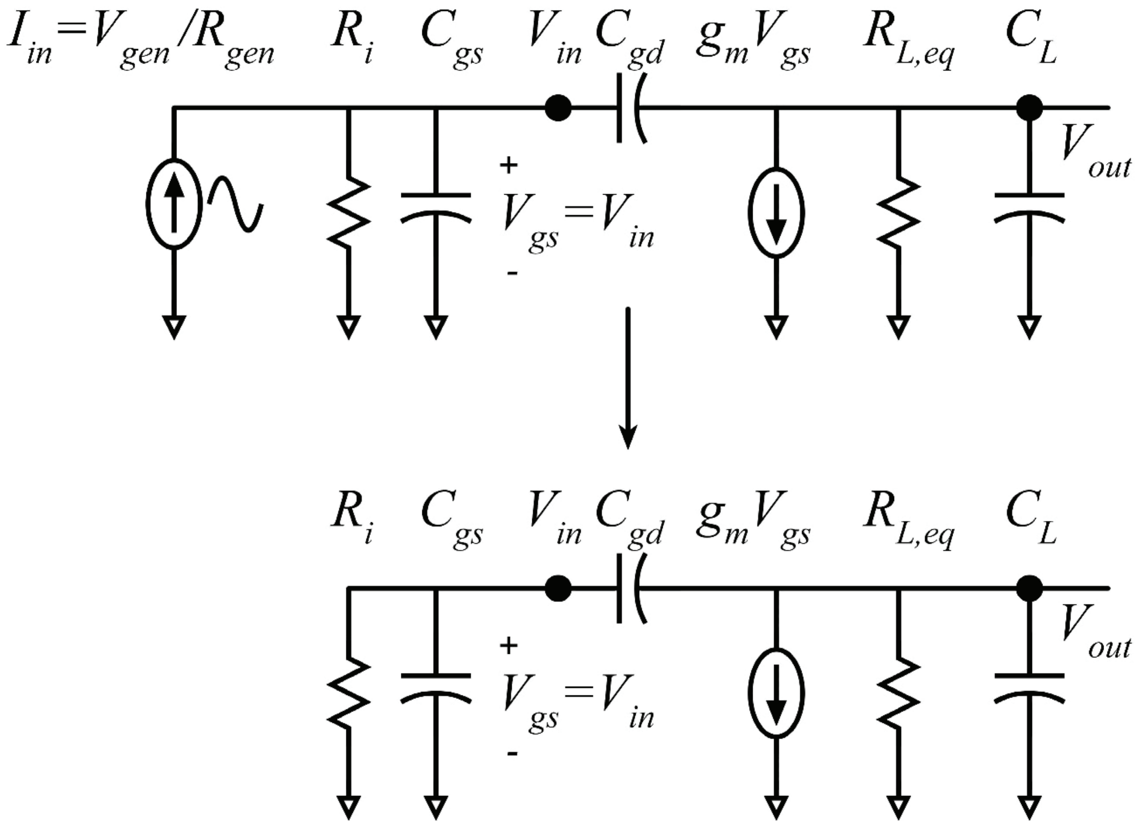


MOTC: Signal sources removed

Turn off independent voltage sources:
i.e. , replace them with short-circuits

Turn off independent current sources:
i.e. , replace them with open-circuits

Dependent (controlled) sources
must be kept.



General Picture

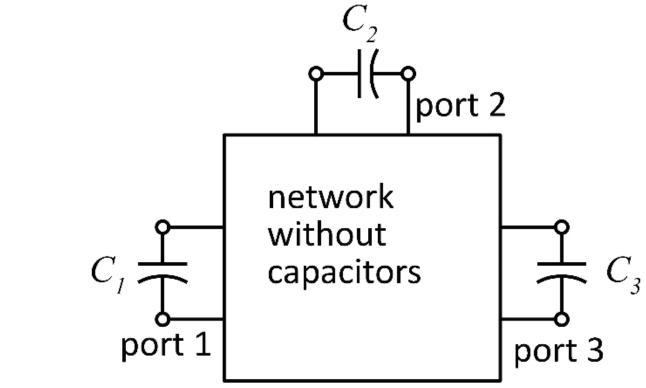
Start with a circuit with resistors, capacitors, controlled sources, but *no inductors*

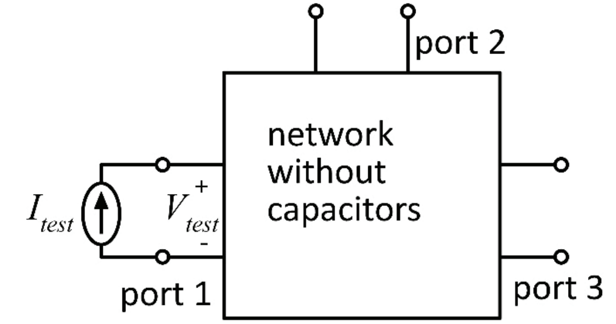
Separate into
1) a frequency-independent circuit resistors and controlled sources, and
2) Capacitors connected to this at ports

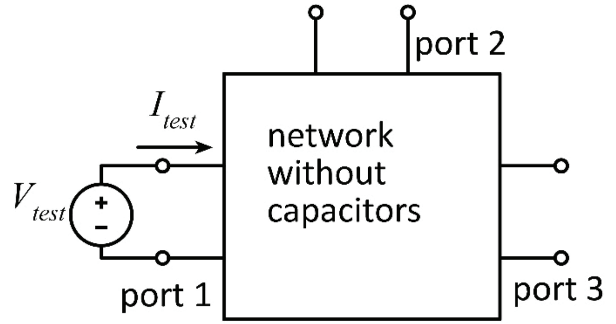
Force a current I_{test} into port 1 and compute V_{test}.

Or apply a voltage V_{test} to port 1 and compute I_{test}.

Define $R_{11}^0 = V_{test} / I_{test}$







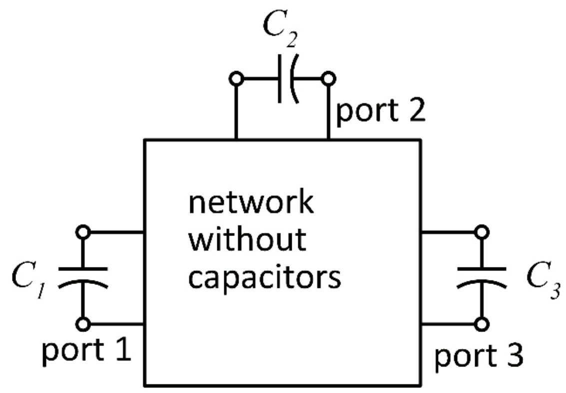
MOTC: first order time constant

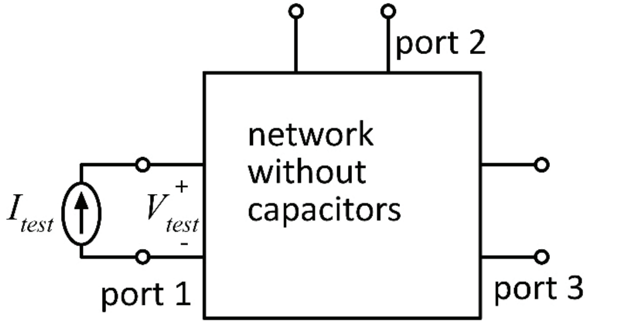
With

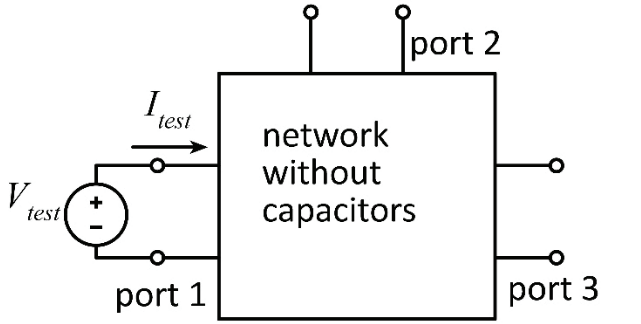
$$\frac{v_{out}(s)}{v_{gen}(s)} = \left.\frac{v_{out}}{v_{gen}}\right|_{\text{mid-band}} \cdot \frac{1 + b_1 s^1 + b_2 s^2 + b_3 s^3 + \ldots}{1 + a_1 s^1 + a_2 s^2 + a_3 s^3 + \ldots}$$

It can be shown that, for an N-capacitor circuit

$$a_1 = R_{11}^0 C_1 + R_{22}^0 C_2 + \ldots. + R_{NN}^0 C_N$$

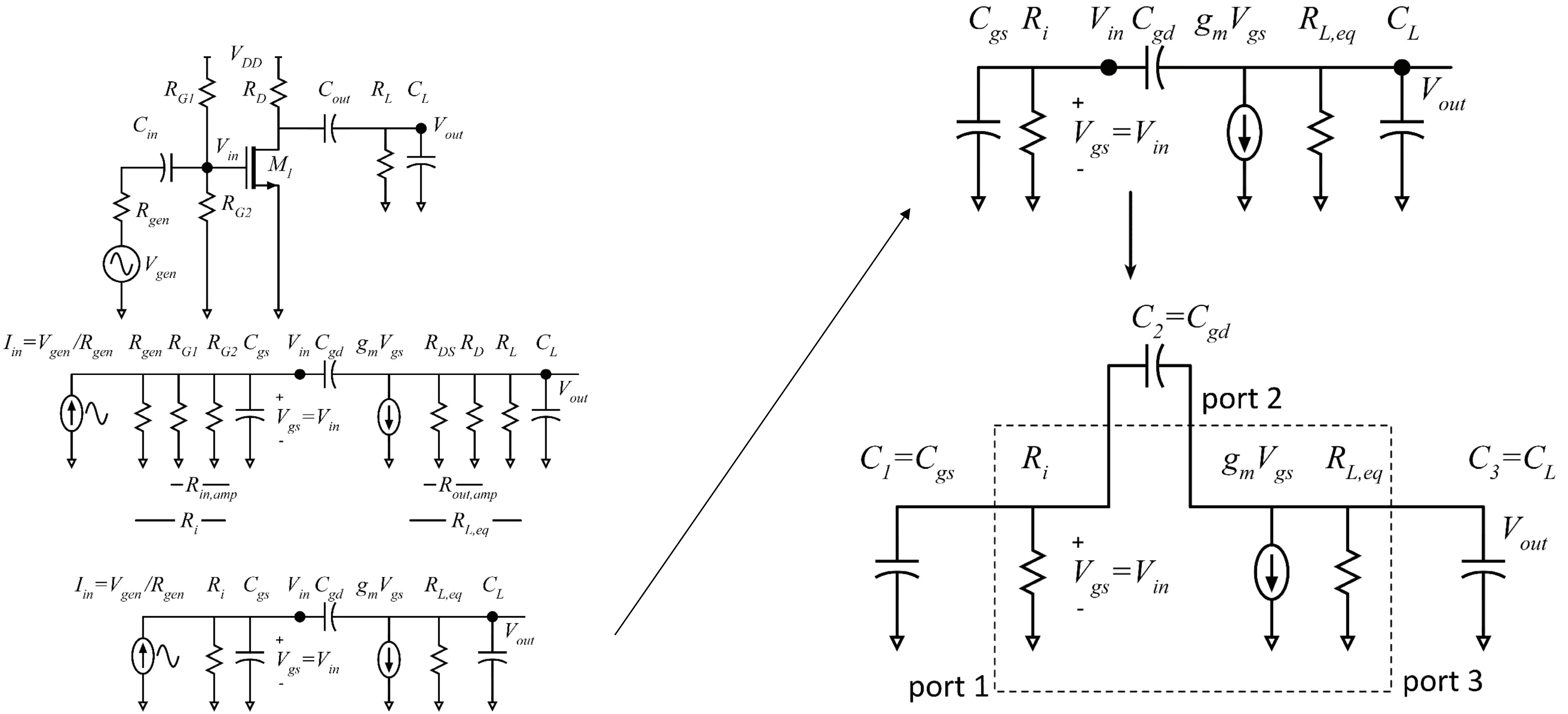






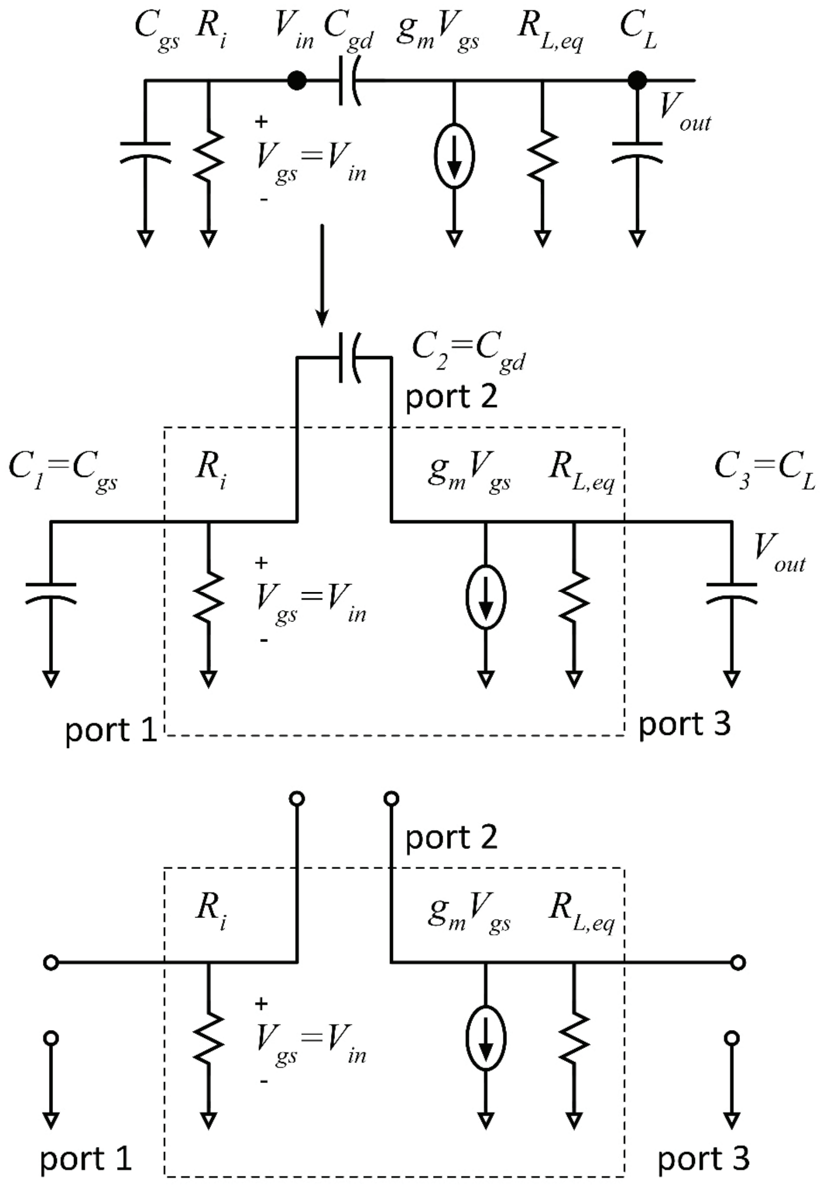
MOTC: Example

For our common-source circuit



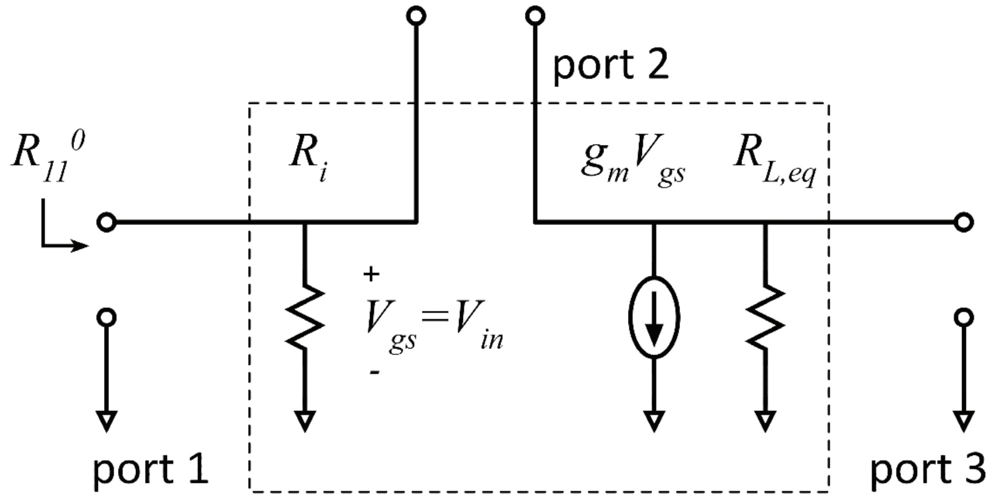
First-order time constants

The capacitor-free 3-port circuit is in the bottom right

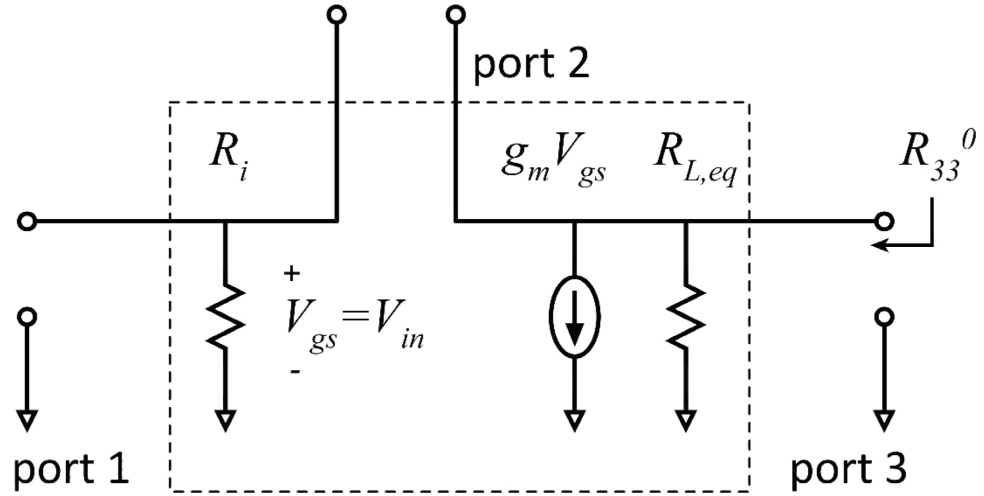


First-order time constants

$R_{11}^{0} = R_i$ (easy) !



$R_{33}^{0} = R_{Leq}$ (easy) !



First-order time constants: The feedback port

To compute R_{22}^0 first use a Norton-Thevenin transformation.

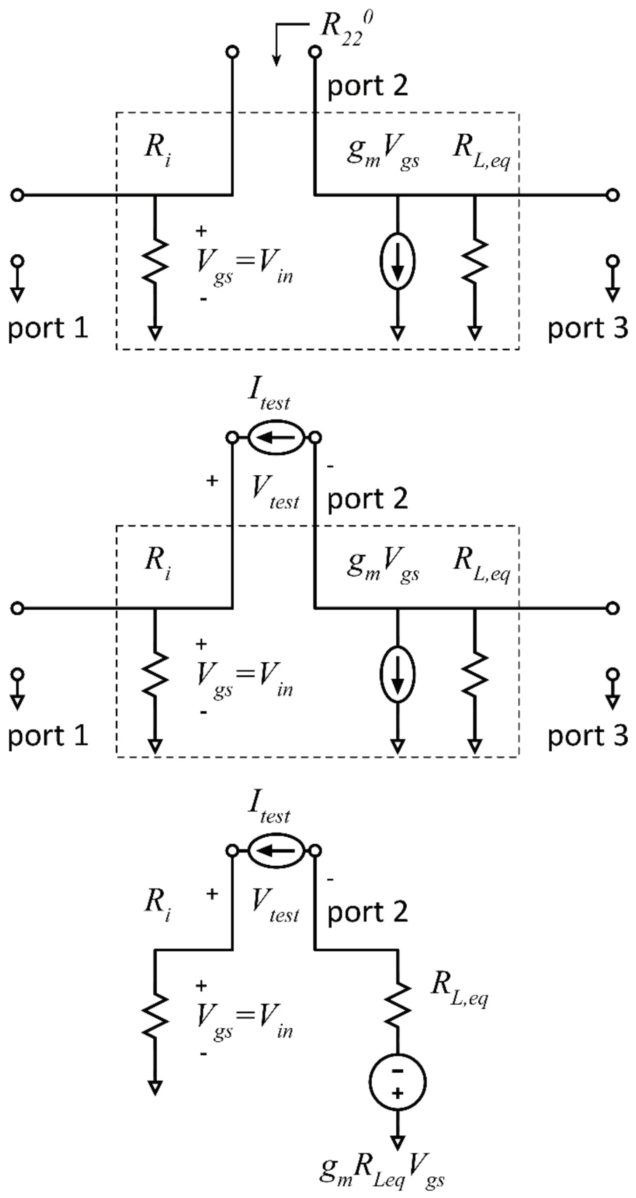
Then $V_{gs} = V_{in} = I_{test} R_i$

and $V_{test} = V_{gs} + g_m R_{Leq} V_{gs} + I_{test} R_{Leq}$

So $V_{test} = I_{test} R_i + g_m R_{Leq} R_i I_{test} + I_{test} R_{Leq}$

So

$$R_{22}^0 = V_{test} / I_{test} = R_i + g_m R_{Leq} R_i + R_{Leq}$$



MOTC and feedback capacitance: General form

We have found that

$$R_{22}^{0} = V_{test} / I_{test} = R_i + g_m R_{Leq} R_i + R_{Leq}$$

But, write instead $R_i = R_{in}$

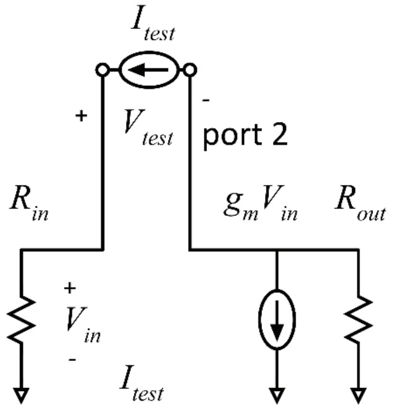
Also note that

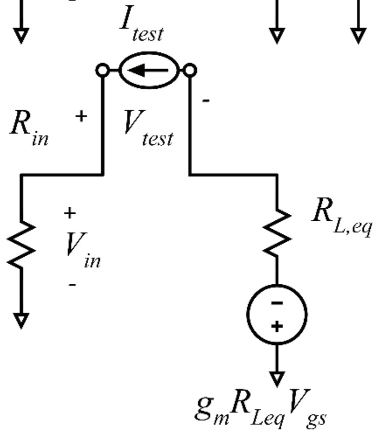
$-A_V = -g_m R_{Leq}$ is the open-circuit voltage gain.

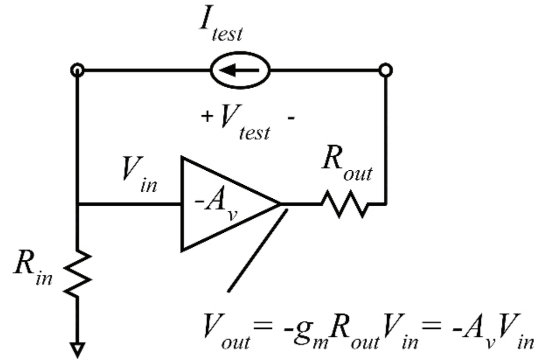
So

$$R_{22}^{0} = R_{in}(1 - A_v) + R_{out}$$

will apply whenever we measure the resistance between the input and output of an amplifier.







MOTC: common-source amplifier

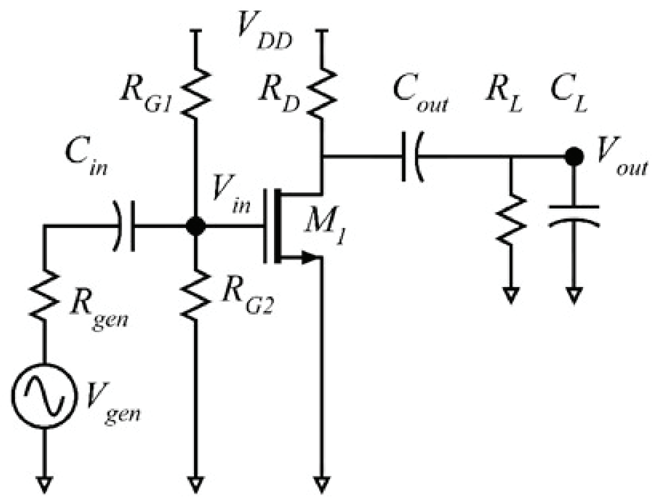
$$R_{11}^0 = R_i$$

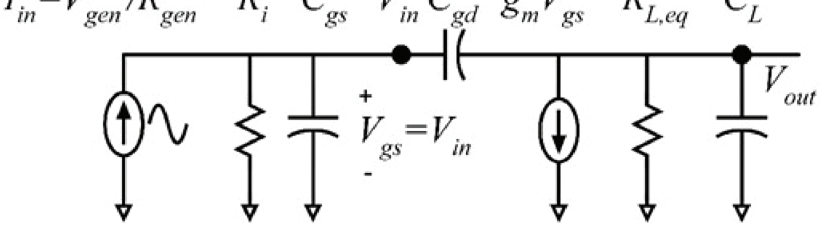
$$R_{22}^0 = R_i(1+g_m R_{Leq}) + R_{Leq}$$

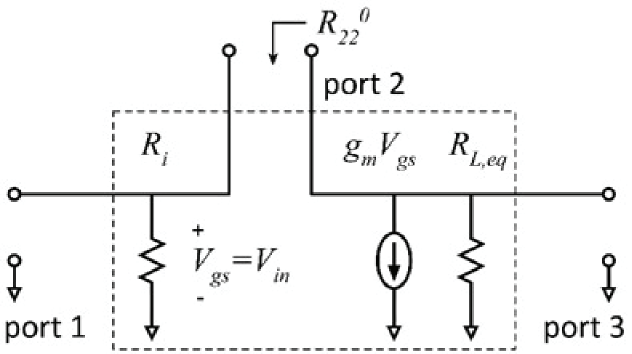
$$R_{33}^0 = R_{Leq}$$

$$a_1 = R_{11}^0 C_1 + R_{22}^0 C_2 + R_{33}^0 C_3$$
$$= R_i C_{gs} + (R_i(1+g_m R_{Leq}) + R_{Leq})C_{gd} + R_{Leq} C_L$$

...this is consistent with nodal analysis







What about the 2nd pole ?

$$\frac{v_{out}(s)}{v_{gen}(s)} = \left.\frac{v_{out}}{v_{gen}}\right|_{\text{mid-band}} \cdot \frac{1+b_1 s^1 + b_2 s^2 + b_3 s^3 + \ldots}{1+a_1 s^1 + a_2 s^2 + a_3 s^3 + \ldots}$$

If a_3 and a_4 etc. are small, then....

$$\frac{v_{out}(s)}{v_{gen}(s)} = \left.\frac{v_{out}}{v_{gen}}\right|_{\text{mid-band}} \cdot \frac{1+b_1 s^1 + b_2 s^2 + b_3 s^3 + \ldots}{1+a_1 s^1 + a_2 s^2}$$

....i.e. we can ignore 3^{rd} and 4^{th} pole due to a_3, a_4.

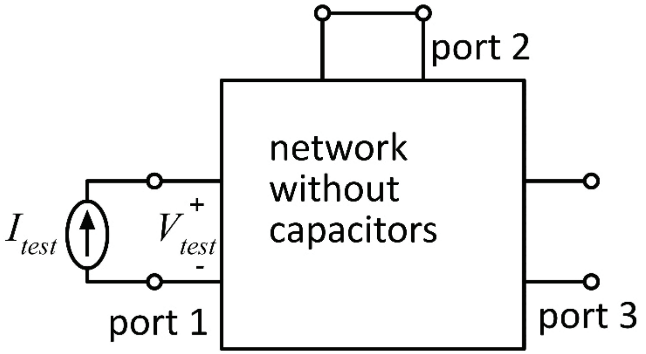
If we can find a_2 , then we can find the 2nd pole.
(again, only if a_3 and a_4 etc. are small)

**** How do we find a_2 ? ****

Defining some more terms:

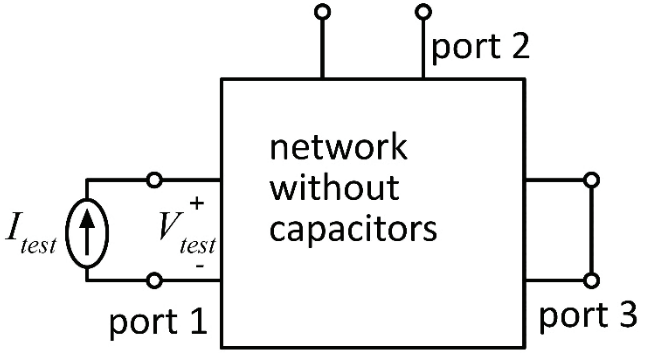
Resistance at port 1 with port 2 shorted.

$$R_{11}^{2} = V_{test} / I_{test}$$



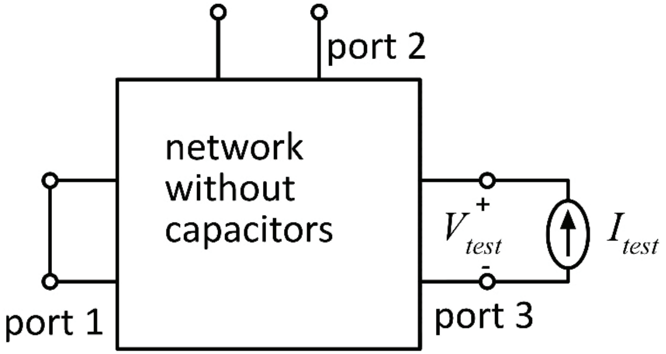
Resistance at port 1 with port 3 shorted.

$$R_{11}^{3} = V_{test} / I_{test}$$



Resistance at port 3 with port 1 shorted.

$$R_{33}^{1} = V_{test} / I_{test}$$



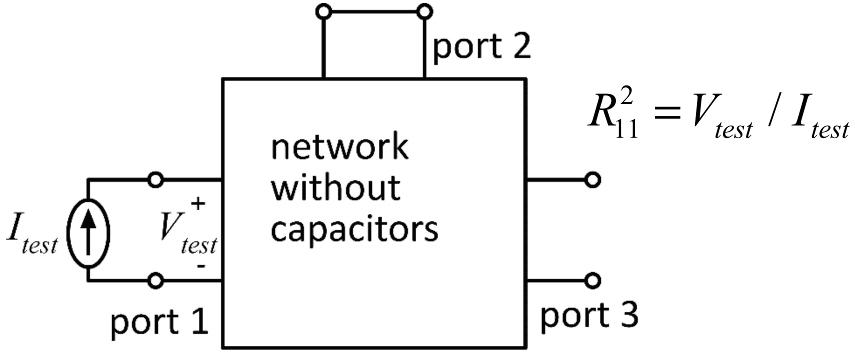
2nd order time constant

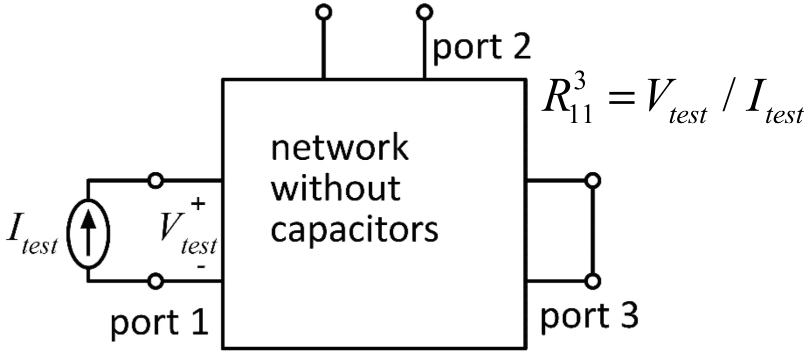
$$\frac{v_{out}(s)}{v_{gen}(s)} = \left.\frac{v_{out}}{v_{gen}}\right|_{\text{mid-band}} \cdot \frac{1+b_1 s^1 + b_2 s^2 + b_3 s^3 + \ldots}{1+a_1 s^1 + a_2 s^2 + a_3 s^3 + \ldots}$$

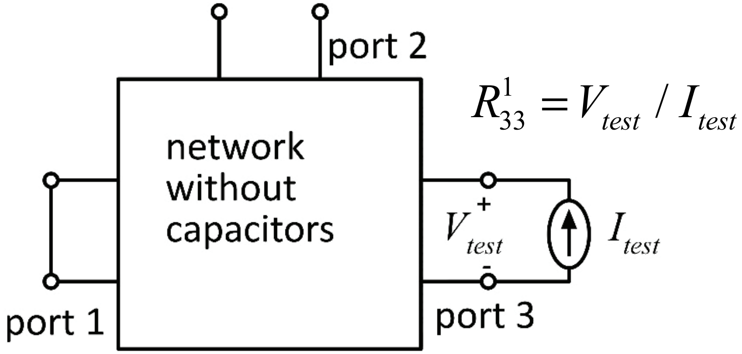
It can also be shown that

$$a_2 = R_{11}^0 C_1 C_2 R_{22}^1 + R_{11}^0 C_1 C_3 R_{33}^1 + R_{11}^0 C_1 C_4 R_{44}^1 + R_{22}^0 C_2 C_3 R_{33}^2 + R_{22}^0 C_2 C_4 R_{44}^2 + R_{33}^0 C_3 C_4 R_{44}^3$$

...answer shown is for 4-capacitor circuit, but it is easy to generalize.

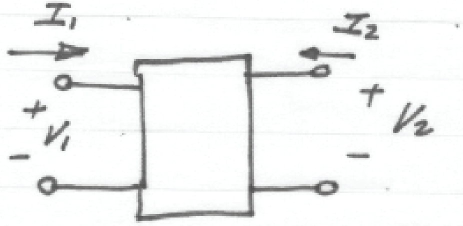






Equivalency of terms.

Now Consider a 2-port for a moment:



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$R_{11}^{0} = Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \quad \& \quad R_{11}^{2} = \left.\frac{V_1}{I_1}\right|_{V_2=0}$$

If $V_2 = 0 \Rightarrow Z_{21} I_1 + Z_{22} I_2 = 0 \Rightarrow \dfrac{V_1}{I_1} = \boxed{Z_{11} + \dfrac{Z_{12}(-Z_{21})}{Z_{22}} = R_{11}^{2}}$

Similarly: $R_{22}^{1} = Z_{22} - \dfrac{Z_{12}Z_{21}}{Z_{11}}$

so: $R_{11}^{0} R_{22}^{1} = Z_{11}\left[Z_{22} - \dfrac{Z_{12}Z_{21}}{Z_{11}}\right] = Z_{11}Z_{22} - Z_{12}Z_{21} = \Delta Z$

and $R_{11}^{2} R_{22}^{0} = \left[Z_{11} - \dfrac{Z_{12}Z_{21}}{Z_{22}}\right] Z_{22} = \ldots = \Delta Z$

so: $\boxed{R_{11}^{0} R_{22}^{1} = R_{22}^{0} R_{11}^{2} = \Delta Z}$

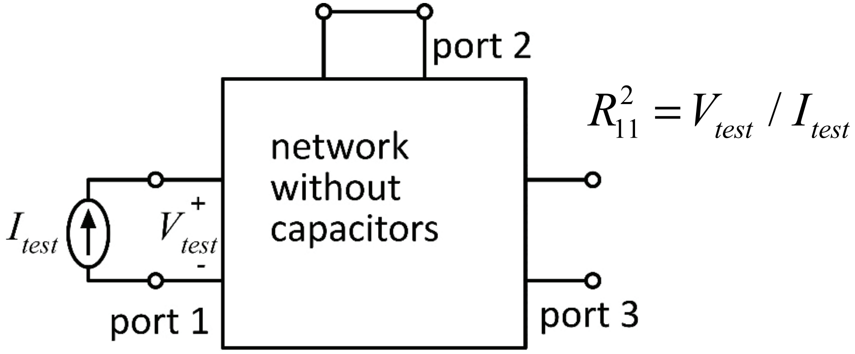
2nd order time constant

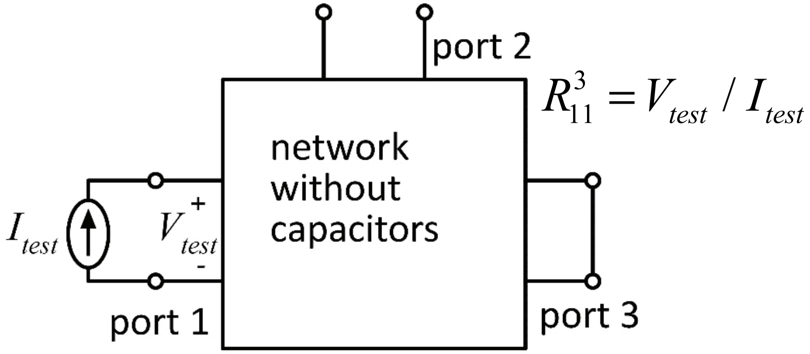
So, in a 3-capacitor circuit:

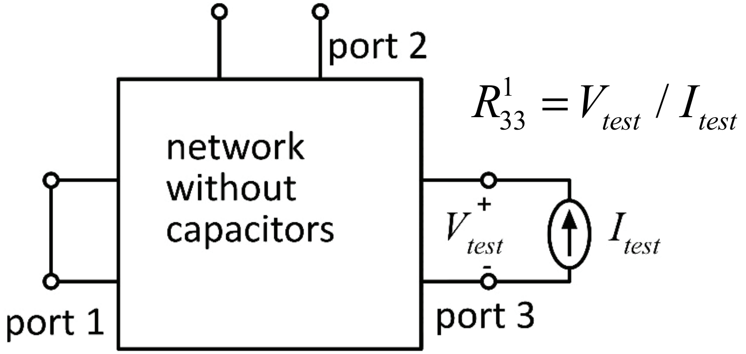
$$a_2 = \left\{ \begin{matrix} R_{11}^0 C_1 C_2 R_{22}^1 \\ \text{or} \\ R_{11}^2 C_1 C_2 R_{22}^0 \end{matrix} \right\} + \left\{ \begin{matrix} R_{11}^0 C_1 C_3 R_{33}^1 \\ \text{or} \\ R_{11}^3 C_1 C_3 R_{33}^0 \end{matrix} \right\} + \left\{ \begin{matrix} R_{11}^0 C_1 C_4 R_{44}^1 \\ \text{or} \\ R_{11}^4 C_1 C_4 R_{44}^0 \end{matrix} \right\}$$

There are always two ways to write each term in a_2.

...answer shown is for 3-capacitor circuit, but it is easy to generalize.







Returning to our example

$C_1 R_{11}^0 R_{22}^1 C_2$:

$R_{11}^0 = R_i$ and $R_{22}^1 = R_{Leq}$

So $C_1 R_{11}^0 R_{22}^1 C_2 = C_{gs} R_i C_{gd} R_{Leq}$

$C_1 R_{11}^0 R_{33}^1 C_3$:

$R_{11}^0 = R_i$ and $R_{33}^1 = R_{Leq}$

So $C_1 R_{11}^0 R_{33}^1 C_3 = C_{gs} R_i C_L R_{Leq}$

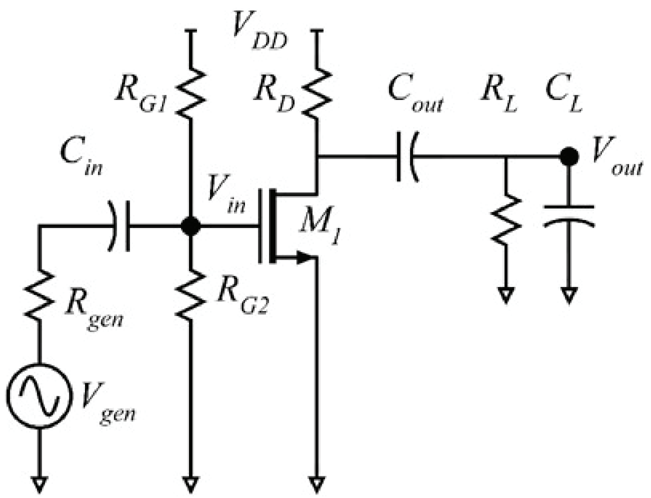
$C_2 R_{22}^0 R_{33}^2 C_3 = C_2 R_{22}^3 R_{33}^0 C_3$:

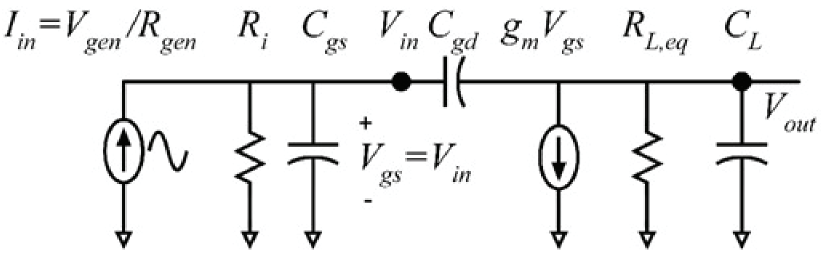
$R_{22}^3 = R_i$ and $R_{33}^0 = R_{Leq}$

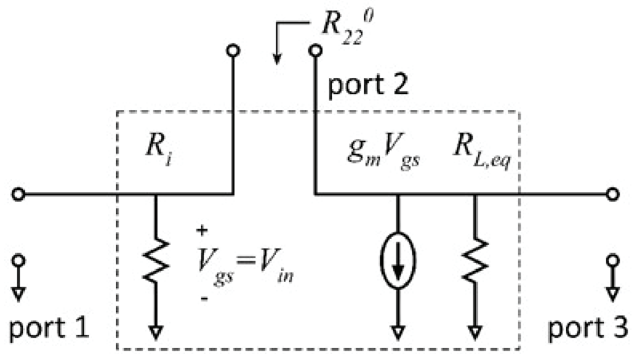
So $C_2 R_{22}^3 R_{33}^0 C_3 = C_{gd} R_i C_L R_{Leq}$

$a_2 = C_{gs} R_i C_{gd} R_{Leq} + C_{gs} R_i C_L R_{Leq} + C_{gd} R_i C_L R_{Leq}$

This is, again, the answer we had found earlier by nodal analysis.







MOTC: Working these Efficiently

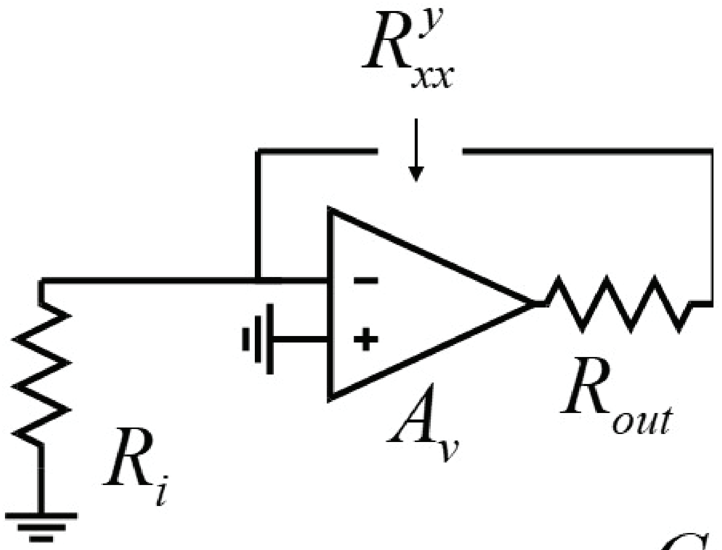
Because $R_{xx}^{y} R_{yy}^{0} = R_{xx}^{0} R_{yy}^{x}$, we always have 2 choices in finding each term in the MOTC.

The trick is to work the
problem so that as much as possible:
1) terms are related to input, ouput, load impedances
2) terms are ones found earlier, in a_1 analysis.

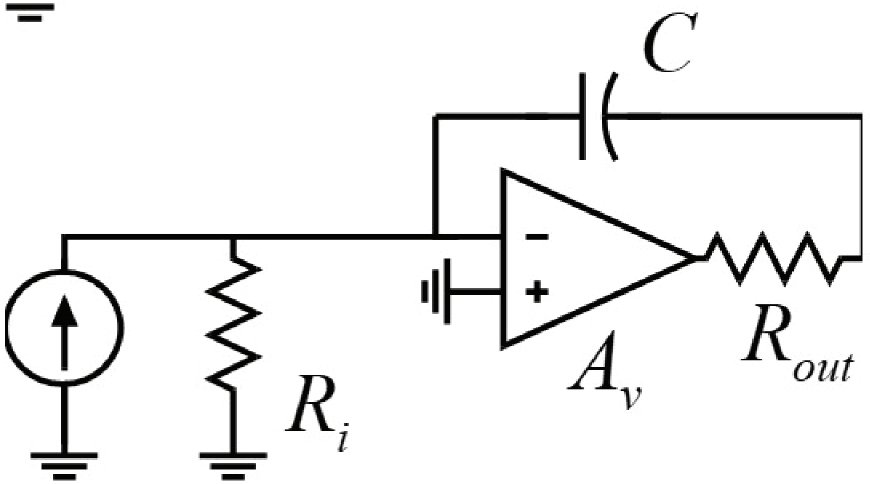
There are 2 "funny" cases which arise so often that I will give them on the next 2 pages (note these are intimately related to the well-known Miller effect)

MOTC and the Miller Effect

$$R_{xx}^{y} = R_i(1+A_v)+R_{out}$$



$$a_1=\tau=\left[R_i(1+A_v)+R_{out}\right]\bullet C$$



MOTC: impedances between collector/drain & base/gate

If we decide explicitly that

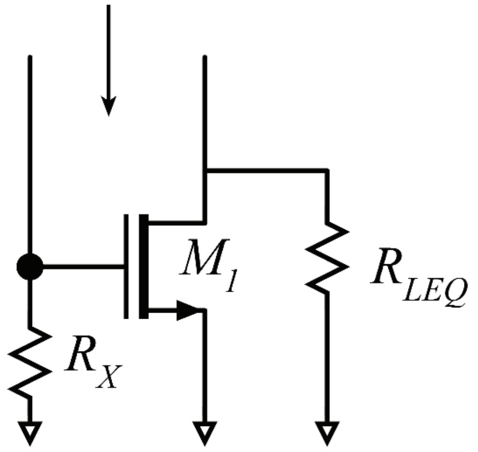
1) R_x is to denote the parallel combination of any external circuit resistances and R_{be},

2) and that R_{Leq} similarly denotes the combined effect of external resistors and R_{ce} or R_{DS},

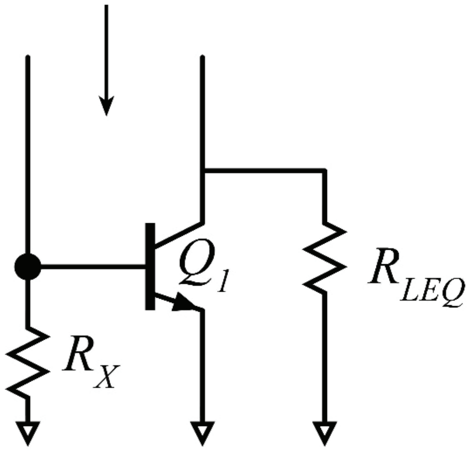
then

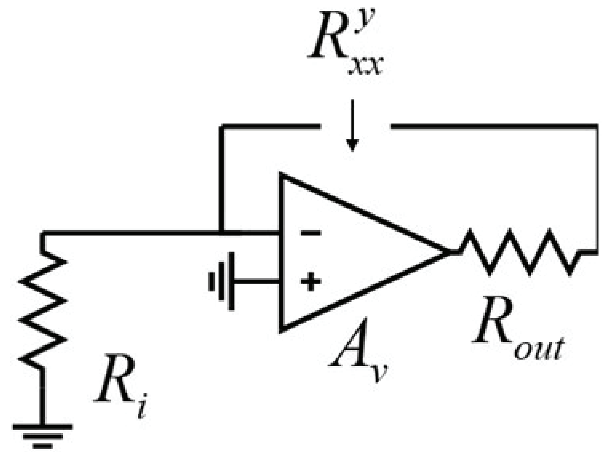
3) $R_{yy}^0 = R_x(1+g_m R_{Leq}) + R_{Leq}$

$R_{yy}^0 = R_x(1+g_m R_{Leq}) + R_{Leq}$



$R_{yy}^0 = R_x(1+g_m R_{Leq}) + R_{Leq}$





MOTC: impedances between emitter/source & base/gate

FET

$R_{yy}^0 = R_i(1 - A_v) + R_{out}$

where $A_v = R_{LEQ} / (R_{LEQ} + g_m^{-1})$

and $R_{out} = R_{LEQ} \,||\, g_m^{-1}$

$$R_{yy}^0 = R_i\left(1 - \frac{R_{LEQ}}{R_{LEQ} + g_m^{-1}}\right) + \frac{g_m^{-1} R_{Leq}}{R_{Leq} + g_m^{-1}} = ...$$

Slightly more complex expressions can be similarly derived for the BJT